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Discontinuity waves cannot describe evanescent waves

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Abstract. The problem of oblique incidence of plane waves on a boundary between two linear media is examined in detail especially in connection with generation of evanescent waves. It is shown that, unlike plane waves, evanescent waves are fully determined by two functions, which are to be chosen as the Hilbert transform of each other so as to guarantee an appropriate behavior of the solution at infinity. When such an approach is contrasted with the theory of discontinuity waves, it turns out that the usual information about discontinuities does not suffice for calculating evanescent waves. In conclusion, the oblique incidence problem does not admit a consistent answer within the sole framework of discontinuity waves.

1. Introduction

The theory of discontinuity wave propagation has attracted many researchers mainly because of the inherent mathematical rigor [1]. However when this theory is applied to the problem of oblique incidence on a boundary – or, even worse, to the problem of the interaction between discontinuity waves and shocks – some unsuspected drawbacks and pitfalls arise. To qualify this claim, we first recall that the general basis for the analysis of the oblique incidence of a discontinuity wave on a boundary in presence of nonlinearities has been dealt with in [2]. Unfortunately, also when the incident wave is a discontinuity wave, it turns out that the theory of discontinuity waves is not appropriate if evanescent waves are generated.

For definiteness, we recall that, essentially, an evanescent wave is a disturbance which propagates along a direction and whose amplitude attenuates along a different direction; it is an interesting result that, for hyperbolic systems, such two directions cannot be parallel [3]. Especially when information on amplitude attenuation is unimportant, harmonic exponential waves of this kind are called inhomogeneous waves; for a thorough analysis of the properties of plane inhomogeneous waves we refer the interested reader to [3–10]. In order to suggest explicitly the account of amplitude attenuation, in accordance with [11], p. 204, we are adopting the term "evanescent waves".

That evanescent waves are relevant for the problem of oblique incidence of plane waves is well known in the literature [12–20]. With the aid of this background material, it is the purpose of this work to arrive at a closed (formal) solution of the oblique incidence problem under the sole assumption that the governing equations take the form of a general linear hyperbolic system. Thus we are able to show that the theory of discontinuity waves is not capable of fully emboding the oblique incidence problem. To make the paper as self-consistent as possible, also part of the background material will be briefly re-elaborated.

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The plane of the paper is as follows. Setting aside nonlinear aspects, a brief résumé of the properties of linear plane waves is presented in Section 2, while Section 3 discusses the general features concerning wave generation by oblique incidence of a plane wave on a boundary. The main properties of evanescent waves are analyzed in Section 4, where it is proved that, unlike plane waves, two functions are needed for determining an evanescent wave. Then, in Section 5, we show that the requirement of well behavior at infinity makes such functions be the Hilbert transform of each other. The comparison with the theory of discontinuity waves is performed in Section 6. As a significant result, we definitively prove not only that the problem of oblique incidence cannot be solved with the sole aid of the discontinuity wave theory but also that the information on the incident discontinuity does not suffice for calculating the possible evanescent waves. Finally, Section 7 is devoted to casting the Stoneley problem [21] within the present framework and to pointing out some obstacles concerning nonlinearities.

2. Plane waves

Consider a physical system whose behavior is described by the following N linear hyperbolic differential equations

$$\frac{\partial U}{\partial t}+A^x\frac{\partial U}{\partial x}+A^y\frac{\partial U}{\partial y}+A^z\frac{\partial U}{\partial z}=0, \tag{2.1}$$

where A^x, A^y, and A^z are constant $N\times N$ matrices.

A plane wave, traveling at speed c along the direction $\mathbf{n}$ ($\mathbf{n}\cdot\mathbf{n}=1$), is a solution to (2.1) which depends on space coordinates $\mathbf{x}$ and time t through the single phase variable

$$\varphi(\mathbf{x},t)=t-\mathbf{n}\cdot\mathbf{x}/c; \tag{2.2}$$

in other words we are looking for solutions of the form

$$U=U(\varphi)=U\left(t-\frac{\mathbf{n}\cdot\mathbf{x}}{c}\right). \tag{2.3}$$

Substitution of (2.3) into (2.1) shows that the possible propagation speeds c coincide with the characteristic speeds relative to the system (2.1), namely with the roots of the secular equation

$$|A_n-cI|=0, \tag{2.4}$$

where I stands for the N dimensional identity matrix and $A_n=A^x\mu+A^y\sigma+A^z\nu$, μ, σ, and ν being the components of $\mathbf{n}$. Also the column vector $U(\varphi)$ can be split as

$$U(\varphi)=\mathscr{U}\left(t-\frac{\mathbf{n}\cdot\mathbf{x}}{c}\right)\Pi, \tag{2.5}$$

where $\mathscr{U}$ is the amplitude while the polarization vector Π is a right eigenvector of the matrix A_n associated with the eigenvalue c, viz a solution to the algebraic system

$$(A_n-cI)\Pi=0. \tag{2.6}$$

Of course both the propagation speed c and the polarization vector Π may depend on the propagation direction $\boldsymbol{n}$ – this dependence has been analyzed, e.g., in [22]; in the sequel we denote by $c(\mathbf{n})$ and $\Pi(\mathbf{n})$ the solutions to (2.4) and (2.6), respectively.

We point out that the wave front of a plane wave (2.3), namely the surface $\varphi(\mathbf{x}, t)=\text{const}$, formally possesses all the features of a discontinuity wave compatible with the system (2.1) – see, e.g., [23]. For propagation problems, indeed, discontinuity waves prove to be more general because they apply as well to differential systems both nonlinear and non-homogeneous. Note, however, that for non-homogeneous linear system, the previous results can be recovered in the limit of waves of infinite frequency [23].

3. Oblique incidence

Let the plane $y=0$ be the boundary between different media; precisely suppose that a medium M, whose behavior is described by the linear hyperbolic system (2.1), occupies the lower half space $y\leqslant 0$, while a different medium $\tilde{M}$, filling the upper half space $y\geqslant 0$, is governed by the linear hyperbolic system

$$\frac{\partial \tilde{U}}{\partial t}+\tilde{A}^x\frac{\partial \tilde{U}}{\partial x}+\tilde{A}^y\frac{\partial \tilde{U}}{\partial y}+\tilde{A}^z\frac{\partial \tilde{U}}{\partial z}=0, \tag{3.1}$$

where $\tilde{A}^x$, $\tilde{A}^y$, and $\tilde{A}^z$ are constant $N\times N$ matrices.

Suppose that a plane wave, traveling into M along the direction $\boldsymbol{n}_{\mathrm{I}}$ at speed $c(\mathbf{n}_{\mathrm{I}})$, impinges on the boundary $y=0$ thereby generating reflected and transmitted waves. Choose the coordinate x so that the plane (x, y) is the plane of incidence; accordingly $\mathbf{n}_{\mathrm{I}}=(\mu_{\mathrm{I}}, \sigma_{\mathrm{I}}, 0)$, $\mu_{\mathrm{I}}>0$, $\sigma_{\mathrm{I}}>0$, with

$$\mu_{\mathrm{I}}^2+\sigma_{\mathrm{I}}^2=1. \tag{3.2}$$

In this case a function $\psi(\mathbf{n}_{\mathrm{I}})$ will be written as $\psi(\mu_{\mathrm{I}}, \sigma_{\mathrm{I}})$. The explicit form of the incident wave is taken to be

$$U^{\mathrm{I}}=\mathscr{U}^{\mathrm{I}}\left(t-\frac{\mu_{\mathrm{I}}x+\sigma_{\mathrm{I}}y}{c(\mu_{\mathrm{I}}, \sigma_{\mathrm{I}})}\right)\Pi(\mu_{\mathrm{I}}, \sigma_{\mathrm{I}}). \tag{3.3}$$

In full generality we assume that p reflected waves and q transmitted waves are generated and are expressed in the form of plane waves as

Reflected waves:

$$U^{\mathrm{R}}=\mathscr{U}^{\mathrm{R}}\left(t-\frac{\mu_R x+\sigma_R y}{c(\mu_R, \sigma_R)}\right)\Pi(\mu_R, \sigma_R), \quad R=1,\ldots,p; \tag{3.4}$$

Transmitted waves:

$$\tilde{U}^T=\tilde{\mathscr{U}}^T\left(t-\frac{\mu_T x+\sigma_T y}{\tilde{c}(\mu_T, \sigma_T)}\right)\tilde{\Pi}(\mu_T, \sigma_T), \quad T=1,\ldots,q. \tag{3.5}$$

In (3.5) the function $\tilde{c}(\mu, \sigma)$ and the column vector $\tilde{\Pi}(\mu, \sigma)$ denote characteristic speeds and the polarization vectors, relative to the medium $\tilde{M}$, expressed in terms of the propagation direction.

As is well known [2, 11, 24], a geometric analysis of the interaction establishes that the emergent modes are determined by Snell's law which states that an emergent wave, traveling along the direction $(\mu, \sigma, 0)$ at speed $c(\mu, \sigma)$, can be generated provided that

$$\frac{\mu_I}{c(\mu_I, \sigma_I)} = \frac{\mu_R}{c(\mu_R, \sigma_R)} = \frac{\mu_T}{\tilde{c}(\mu_T, \sigma_T)}. \tag{3.6}$$

Note that the quantity σ is subjected to the only constraint (3.2); hence the reflection-transmission pattern is not unique. To restore uniqueness we must impose the requirement of causality; to this purpose the literature bears evidence of different statements concerning causality – see, e.g., [2, 21, 24]. Things can be even more complicated when evanescent waves are present [25]; therefore we shall appeal to causality as soon as we need it.

So as to evaluate the amplitudes of the emergent waves determined by Snell's law (3.6), we remark that both systems (2.1) and (3.1) are conservative and that the boundary $y=0$ acts as a strong discontinuity for the field variables. This leads us to assume that the fields on both sides are connected by the generalized Rankine–Hugoniot conditions [26]. The conservative form for the system (2.1) is

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}(A^x U) + \frac{\partial}{\partial y}(A^y U) + \frac{\partial}{\partial z}(A^z U) = 0,$$

hence, the generalized Rankine–Hugoniot conditions establish that, for $y=0$,

$$A^y U^I + \sum_{R=1}^{p} A^y U^R = \sum_{T=1}^{q} \tilde{A}^y \tilde{U}^T. \tag{3.7}$$

In view of (3.3)–(3.5) condition (3.7) becomes

$$\sum_{R=1}^{p} A^y \Pi(\mu_R, \sigma_R) \mathscr{U}^R(\varphi_0) - \sum_{T=1}^{q} \tilde{A}^y \tilde{\Pi}(\mu_T \sigma_T) \tilde{\mathscr{U}}^T(\varphi_0) + A^y \Pi(\mu_I, \sigma_I) \mathscr{U}^I(\varphi_0) = 0; \tag{3.8}$$

note that, owing to Snell's law (3.6), the argument of all amplitudes coincide with the quantity $\varphi_0 = t - x\mu_I / c(\mu_I, \sigma_I)$.

Suppose now that Snell's law (3.6) gives rise only to real values of μ subject to the condition $\mu \leqslant 1$; then we say that all the emergent waves are real. Accordingly, the causality condition can be stated mathematically in the following way: on assuming that the propagation speeds are positive, we must choose [2]

$$\sigma_R = -\sqrt{1-\mu_R^2} < 0, \qquad \sigma_T = \sqrt{1-\mu_T^2} > 0.$$

As to the amplitudes of the emergent waves, we must suppose also that the system (3.8) admits a unique solution for the $p+q$ quantities $\mathscr{U}^R$ and $\tilde{\mathscr{U}}^T$. Note, however, that this is not the case for grazing incidence in linear elastic crystals [27].

4. Evanescent waves

So far all the emergent waves are characterized by a quantity μ which was assumed to be real and such that $\mu \leqslant 1$. Accordingly the quantity σ, calculated from (3.2) was both real and $\sigma \leqslant 1$. However, Snell's law often provides us with values of μ which can be either $\mu > 1$ or even a complex quantity [28]. In both cases σ turns out to be a complex quantity and the corresponding waves do not travel at a characteristic speed [3].

To make the analysis more compact, we suppose that the quantity μ, calculated via Snell's law, takes on a complex value. Note first that whenever μ becomes a complex number – or even when $\mu > 1$ – the polarization vectors may involve complex quantities as a consequence of (2.6); in turn the Rankine–Hugoniot conditions (3.7) involve complex quantities thereby making the amplitudes complex too. The presence of complex quantities renders their meaning not immediate. The guiding idea is that of employing such formal complex solutions as a tool for finding explicitly the real evanescent solutions we are looking for.

We alert the reader that the following use of the complex formalism is completely different from the very common technique of representing the plane wave solutions in a complex exponential form and stipulating that only the real or imaginary part has the actual physical significance. Here, instead, it is the very structure of the problem which makes complex quantities unavoidable: our task is that of profiting of such a peculiarity for calculating the real evanescent solutions.

To fix notation we denote the real and imaginary part of a complex quantity ψ in accordance with the formula

$$\psi = \psi_1 + \mathrm{i}\psi_2, \quad \mathrm{i} = \sqrt{-1}.$$

First, look at the phase (2.2). It is a consequence of Snell's law that the ratio between μ and the corresponding propagation speed $c = c(\mu, \sigma)$ is always a real quantity and that

$$\frac{\mu}{c} = \frac{\mu_1}{c_1} = \frac{\mu_2}{c_2} \in \mathbb{R}.$$

Accordingly, the phase variable φ takes the formal expression

$$\varphi = t - \frac{\mu_1}{c_1} x - \alpha y,$$

where the quantity α is a shorthand for

$$\alpha = \frac{\sigma}{c} = \frac{c_1\sigma_1 + c_2\sigma_2}{c_1^2 + c_2^2} + \mathrm{i}\,\frac{c_1\sigma_2 - c_2\sigma_1}{c_1^2 + c_2^2}.$$

In term of the new variables τ and Y, defined by

$$\tau = t - \frac{\mu_1}{c_1} x - \alpha_1 y, \tag{4.1}$$

$$Y = \alpha_2 y, \tag{4.2}$$

we have

$$\varphi = \tau - \mathrm{i}Y. \tag{4.3}$$

In view of (2.3), the appearance of a complex phase (4.3) is here simply interpreted as the suggestion that we look for a (real) solution to the system (2.1) in the form $U = U(\tau, Y)$. Substitution into (2.1) shows that the function $U(\tau, Y)$ must satisfy the system

$$A^{\tau}\frac{\partial U}{\partial \tau} + A^{Y}\frac{\partial U}{\partial Y} = 0, \tag{4.4}$$

where

$$A^\tau = I - \frac{\mu_1}{c_1} A^x - \alpha_1 A^y, \tag{4.5}$$

$$A^Y = \alpha_2 A^y. \tag{4.6}$$

So as to find a solution to the system (4.4), we can profitably employ the presence of complex quantities in the following way. As already noticed, the polarization vector Π may become complex. Owing to (4.5) and (4.6), condition (2.6) implies that the real and imaginary part of the polarization vector satisfy the system

$$A^\tau \Pi_1 + A^Y \Pi_2 = 0, \qquad A^Y \Pi_1 - A^\tau \Pi_2 = 0. \tag{4.7}$$

Consider now formally the expression (2.5) where the (possibly complex) function $\mathscr{U}$ is arbitrary and the phase φ is given by (4.3). Since

$$\mathscr{U}(\tau - \mathrm{i}Y) = \mathscr{U}_1(\tau, Y) + \mathrm{i}\mathscr{U}_2(\tau, Y),$$

we can write (2.5) in the form

$$U = (\mathscr{U}_1 + \mathrm{i}\mathscr{U}_2)(\Pi_1 + \mathrm{i}\Pi_2) = (\mathscr{U}_1 \Pi_1 - \mathscr{U}_2 \Pi_2) + \mathrm{i}(\mathscr{U}_1 \Pi_2 + \mathscr{U}_2 \Pi_1). \tag{4.8}$$

It is a straightforward matter to ascertain that, whatever function $\mathscr{U}$ we choose, the real and imaginary part of (4.8) separately satisfy the system (4.4). To this end, consider the real part of (4.8). Substitution into (4.4) yields

$$A^\tau \Pi_1 \left(\frac{\partial \mathscr{U}_1}{\partial \tau} + \frac{\partial \mathscr{U}_2}{\partial Y} \right) - A^\tau \Pi_2 \left(\frac{\partial \mathscr{U}_2}{\partial \tau} - \frac{\partial \mathscr{U}_1}{\partial Y} \right) = 0,$$

which is satisfied in view of the Cauchy–Riemann conditions

$$\frac{\partial \mathscr{U}_1}{\partial \tau} + \frac{\partial \mathscr{U}_2}{\partial Y} = 0, \qquad \frac{\partial \mathscr{U}_2}{\partial \tau} - \frac{\partial \mathscr{U}_1}{\partial Y} = 0,$$

for the complex function $\mathscr{U}(\tau - \mathrm{i}Y)$. An analogous result holds for the imaginary part of (4.8). Owing to the linearity of (4.4), we conclude that the real solution $U = U(\tau, Y)$ is expressed as a superposition of the real and imaginary part of (4.8).

The present task is now that of applying this result to the problem of the oblique incidence by determining the amplitude of the emergent waves through the use of the Rankine–Hugoniot conditions (3.7). Of course, the Rankine–Hugoniot conditions must be satisfied by the real solution $U = U(\tau, Y)$; here, however, we insist on taking advantage of the complex formalism by going back to (4.8). Accordingly, the Rankine–Hugoniot conditions (3.7) split as

$$\begin{cases} \sum\limits_{R=1}^{p} A^y (\mathscr{U}_1^R \Pi_1^R - \mathscr{U}_2^R \Pi_2^R) - \sum\limits_{T=1}^{q} \tilde{A}^y (\tilde{\mathscr{U}}_1^T \tilde{\Pi}_1^T - \tilde{\mathscr{U}}_2^T \tilde{\Pi}_2^T) + A^y \mathscr{U}^{\mathrm{I}} \Pi^{\mathrm{I}} = 0, \\ \sum\limits_{R=1}^{p} A^y (\mathscr{U}_1^R \Pi_2^R + \mathscr{U}_2^R \Pi_1^R) - \sum\limits_{T=1}^{q} \tilde{A}^y (\tilde{\mathscr{U}}_1^T \tilde{\Pi}_2^T + \tilde{\mathscr{U}}_2^T \tilde{\Pi}_1^T) = 0. \end{cases} \tag{4.9}$$

Since the solution to the system (4.9) will depend linearly on the amplitude $\mathscr{U}^{\mathrm{I}}$ of the incident wave, we set

$$\mathscr{U}^R = k^R \mathscr{U}^{\mathrm{I}}, \qquad \tilde{\mathscr{U}}^T = k^T \mathscr{U}^{\mathrm{I}}, \tag{4.10}$$

where the constants k^R and k^T are complex quantities. By substituting (4.10) into (4.9) we arrive at the following algebraic system

$$\begin{cases} \sum_{R=1}^{p} A^y(\Pi_1^R k_1^R - \Pi_2^R k_2^R) - \sum_{T=1}^{q} \tilde{A}^y(\tilde{\Pi}_1^T k_1^T - \tilde{\Pi}_2^T k_2^T) + A^y \Pi^{\mathrm{I}} = 0, \\ \sum_{R=1}^{p} A^y(\Pi_2^R k_1^R + \Pi_1^R k_2^R) - \sum_{T=1}^{q} \tilde{A}^y(\tilde{\Pi}_2^T k_1^T + \tilde{\Pi}_1^T k_2^T) = 0. \end{cases} \tag{4.11}$$

which allows us to determine the $2(p+q)$ quantities k_1^R, k_2^R, k_1^T, and k_2^T. We explicitly assume that the system (4.11) admits a unique solution.

As already remarked, the general solution can be taken as a superposition (with real coefficients) of the real and imaginary part of (4.8). However, the real part satisfies $(4.9)_1$ which involves the amplitude of the incident wave, whereas the imaginary part satisfies $(4.9)_2$ which is homogeneous. Hence, in this latter case, the amplitude of the incident wave does not play any privileged role; accordingly, the imaginary part of (4.8) can be determined through relations (4.10) by substituting the incident amplitude $\mathscr{U}^{\mathrm{I}}$ with an arbitrary function $\mathscr{V}^{\mathrm{I}}$. In conclusion the general form of the emergent waves can be calculated as follows. For every reflected or transmitted wave, define the quantities $\mathscr{U}$ and $\mathscr{V}$ by the formulae

$$\mathscr{U} = (k_1 + \mathrm{i}k_2)(\mathscr{U}_1^{\mathrm{I}} + \mathrm{i}\mathscr{U}_2^{\mathrm{I}}), \qquad \mathscr{V} = (k_1 + \mathrm{i}k_2)(\mathscr{V}_1^{\mathrm{I}} + \mathrm{i}\mathscr{V}_2^{\mathrm{I}}), \tag{4.12}$$

where the constants k_1 and k_2 are the relevant values calculated by solving the algebraic system (4.11). Then every emergent wave takes the form

$$U = (\mathscr{U}_1 + \mathscr{V}_2)\Pi_1 - (\mathscr{U}_2 - \mathscr{V}_1)\Pi_2. \tag{4.13}$$

It is apparent that, when evanescent waves are generated, the solution depends on the amplitude $\mathscr{U}^{\mathrm{I}}$ of the incident wave and on an arbitrary function $\mathscr{V}^{\mathrm{I}}$ which will be chosen by imposing the regularity of the solution.

5. Hilbert transforms

In view of (4.12), the solution (4.13) is the real part of a suitable complex N-dimensional column vector, precisely

$$U = \Re[(\mathscr{U}^{\mathrm{I}} - \mathrm{i}\mathscr{V}^{\mathrm{I}})k\Pi], \tag{5.1}$$

where the values of the complex number k, solutions to (4.11), single out the specific emergent wave. In accordance with formula (5.1), the regularity of every emergent wave, in the pertinent half space of definition, is accounted for through the regularity of the complex quantity $\mathscr{U}^{\mathrm{I}} - \mathrm{i}\mathscr{V}^{\mathrm{I}}$, as function of $\tau - \mathrm{i}Y$. Fortunately, problems of this kind have been solved in the literature by having recourse to the Hilbert transforms. Specifically, the Hilbert transform $g(x)$ of a function $f(x) \in L^2(-\infty, \infty)$ is defined by the formula

$$g(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(\chi)}{\chi - x} \, \mathrm{d}\chi,$$

P denoting a principal value at $\chi = x$. In [29], p. 128, the following is proved.

Theorem. *Let $z=x+\mathrm{i}y$ and let $\Phi(x)$ be a complex function of the real variable x belonging to $L^2(-\infty, \infty)$. Then $\Phi(x)$ is the limit as $z \to x$ of an analytic function $\Phi(x)$, regular for $y>0$, such that*

$$\int_{-\infty}^{\infty} |\Phi(x+\mathrm{i}y)|^2 \,\mathrm{d}x < K,$$

K being a positive constant, if and only if $\Phi(x)=f(x)-\mathrm{i}g(x)$, where $f(x)$ and $g(x)$ are the Hilbert transforms of each other.

Suppose that $\mathscr{U}^{\mathrm{I}}$ belongs to $L^2(-\infty, \infty)$. A proper application of this result establishes that the choice of the function $\mathscr{V}^{\mathrm{I}}$ as the Hilbert transform of the amplitude $\mathscr{U}^{\mathrm{I}}$ of the incident wave makes the complex function $\mathscr{U}^{\mathrm{I}}-\mathrm{i}\mathscr{V}^{\mathrm{I}}$ into an analytic function of the complex variable $\tau-\mathrm{i}Y$ which is regular in the domain $Y<0$. Consequently, the regular behavior of the emergent waves results in a restriction on the sign of the quantity α_2 appearing in (4.2): reflected waves, defined in the lower half space, require $\alpha_2>0$, whereas transmitted waves, traveling in the upper half space, force $\alpha_2<0$.

We emphasize that the physical requirement of a regular behavior of the emergent waves also restores the uniqueness of the solution because the function $\mathscr{V}^{\mathrm{I}}$ is uniquely selected by the necessary and sufficient condition of the previous theorem. This reciprocal link between regularity at infinity and uniqueness is peculiar of the presence of an evanescent wave. As a matter of face, uniqueness of real waves do not imply any regularity hypothesis on the amplitude $\mathscr{U}^{\mathrm{I}}$ of the incident wave.

Plane exponential waves follow at once from our general approach by noticing that $\cos\xi$ and $-\sin\xi$ are conjugate functions [29]. Thus, our results provide a further motivation for the convenient use of the complex exponential function.

6. The problem of discontinuity waves

As already remarked at the end of Section 2, in the case of linear homogeneous systems the surfaces of constant phase are mathematically equivalent to discontinuity waves and even to shocks. Specifically, as discontinuities do [30], such surfaces travel at a characteristic speed, determined by the determinantal condition (2.4), and the polarization vectors, solutions to the algebraic system (2.6), are exactly the right eigenvectors of the standard theory of discontinuity waves [31]. The very difference between plane waves and discontinuity waves is that a plane wave is fully determined when its amplitude, as a function of the phase, is known; on the contrary, discontinuity waves do not require such a detailed information. Indeed their equivalence stems from the circumstance that, for linear systems, the amplitude of a plane wave does not play any role as far as its propagation is concerned.

Differently, evanescent waves select two distinguished planes,

$$\tau=\text{const}, \qquad Y=\text{const},$$

which do not depend on the detailed expression (4.13). Of course, the explicit form (4.13) can be arrived at by the procedure exhibited in Section 4, which only requires the choice of a regular (complex) function $\mathscr{U}$ of a single variable. Also, evanescent waves always travel at a speed which is not characteristic – see [3]. As a remarkable consequence, evanescent waves never represent discontinuities.

This paper examined a physical problem where evanescent waves are of vital importance, namely the oblique incidence of a plane wave on a boundary. Indeed, Section 4 and 5 show that calculation of the amplitudes of the emergent waves requires the account of possible evanescent waves. Owing to the equivalence between plane waves and discontinuities, the same is true when the incident wave is a discontinuity wave. Unfortunately, in this latter case the discontinuity wave does not provide any information on the amplitude $\mathscr{U}^{\mathrm{I}}$ of the associated plane wave; therefore it is not possible to determine the explicit form (4.13) of the emergent evanescent waves. Accordingly, we draw the conclusion that such kind of problems cannot be solved within the sole framework of discontinuity waves, as announced at the very beginning.

7. Final remarks

The general features of oblique incidence of a plane wave on an interface between two linear media have been analyzed in detail. Now we aim at proving that it is also possible to frame the Stoneley problem [21] in this context. Roughly speaking, the Stoneley problem can be viewed as an oblique incidence problem where no incident wave is present. With this interpretation, the relevant solution to (2.1) is assumed to be of the form $U(\tau, Y)$ where now we have

$$\tau = t - \lambda x - \alpha_1 y, \qquad Y = \alpha_2 y,$$

where λ, α_1, and α_2 are real quantities to be determined. Conditions on such quantities follow from the solvability of the linear systems (4.7) and (4.9), with $\mathscr{U}^{\mathrm{I}} = 0$. The Stoneley problem admits a solution of the form (4.13) provided that the solvability conditions are compatible with real values of λ, α_1, and α_2.

Finally, we briefly discuss the important case of nonlinear evolution systems to point out a few intrinsic difficulties. In essence, nonlinearity plays a twofold role. First, calculation of evanescent waves is not so clear and straightforward also because of the lack of a sort of Hilbert transform technique valid for nonlinear problems. Second, genuine nonlinearity, in the Lax sense [32], makes the amplitude of a discontinuity wave blow up in a finite time, called the critical time. Accordingly, when the incident wave is a discontinuity wave, its amplitude, as well as the amplitude of the emergent waves, can blow up in a finite time thereby rendering the problem intrinsically ill posed. It seems that this point has been overlooked in the literature [2, 24, 33]. Although of formidable difficulty, we believe that the problem of oblique incidence in presence of nonlinearity deserves further special attention and analysis.

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