High velocity frame transformations
I: Projection operators

FRANCO BAMPI
Dipartimento di Ingegneria della Produzione, Termoenergetica e Modelli Matematici
University of Genova
16129 Genova, Italy
bampi@diptem.unige.it

CLARA ZORDAN
Dipartimento di Ingegneria Biofisica ed Elettronica
University of Genova
16145 Genova, Italy
zordan@dibe.unige.it

Abstract: *Frame transformations between inertial observers can be conveniently analyzed by introducing a suitable set of projection operators, which represent an useful formalism for calculations. The properties of these operators are investigated in details. Finally it is shown that the condition of rotational invariance of a matrix around a fixed direction is described by the aid of such operators.*

1. Introduction

The search for a transformation between inertial observers compatible with high speed experiments (such as the celebrated Michelson-Morley experiment) led Einstein to formulate the special theory of relativity. As a matter of fact, Einstein made also another assumption: that the light speed is constant along open paths. Since then a number of papers appeared for discussing whether special relativity is the unique theory which agrees with the experimental results. An answer to this problem was provided also by ourselves in ref. [1] where it was shown that there is a class of frame transformations, parameterized by an arbitrary three-vector, which account for both Michelson-Morley experiment and transverse Doppler effect. A thorough discussion of this problem, completed also by a historical survey, was carried out by Mansouri and Sexl in ref. [2].

These two papers approach different problems: we were interested in deducing the most general frame transformation compatible with accounting for some relevant experiments, whereas Mansouri and Sexl looked for a general theory, obtained by imposing some "natural" kinematical conditions, by means of which to test special relativity.

In this series of paper, we compare critically the two approaches with a particular attention to the problem of clock synchronization in a genuine three dimensional case. This first paper is devoted to develop a projection operator formalism which turns out to be very useful for the subsequent calculations. Also, the rotational invariance of a matrix around a fixed direction is proved to be conveniently described by the aid of such operators.

2. Projection operators

In the approach to frame transformations the velocity $\mathbf{v}$ between observers plays the role of a distinguished vector singled out, in a natural way, by the two frames. This leads us to introduce some projection operators and analyze their properties.

Precisely define the parallel projector $\mathbf{\Pi}$ and the orthogonal projector $\mathbf{\Omega}$ as

(2.1) $$\Pi^i{}_j = \frac{v^i v_j}{v^2}$$

(2.2) $$\Omega^i{}_j = \delta^i{}_j - \frac{v^i v_j}{v^2}$$

where $v^2 = \mathbf{v}\cdot\mathbf{v}$ and latin indices run from 1 to 3. Of course they are symmetric and behave as a complete set of projectors, i.e.

(2.3) $$\Pi^i{}_k \Pi^k{}_j = \Pi^i{}_j \ ; \quad \Omega^i{}_k \Omega^k{}_j = \Omega^i{}_j \ ; \quad \Pi^i{}_k \Omega^k{}_j = \Omega^i{}_k \Pi^k{}_j = 0 \ ; \quad \Pi^i{}_j + \Omega^i{}_j = \delta^i{}_j$$

as it is easy to prove by direct calculation. In accordance with their definition, for every vector $\mathbf{a}$ we set

(2.4) $$a^i_\parallel = \Pi^i{}_j a^j \quad \Leftrightarrow \quad \mathbf{a}_\parallel = \mathbf{\Pi}\,\mathbf{a}$$

(2.5) $$a^i_\perp = \Omega^i{}_j a^j \quad \Leftrightarrow \quad \mathbf{a}_\perp = \mathbf{\Omega}\,\mathbf{a}$$

For further reference – see ref. [3] – it is convenient to split $\mathbf{\Omega}$ into two other projectors in the following way. First of all we denote with $\mathbf{P(u)}$ the projection operator along the direction of the unit vector $\mathbf{u}$, that is to say $P(\mathbf{u})^i{}_j = u^i u_j$. Accordingly we could write $\mathbf{\Pi} = \mathbf{P}(\mathbf{v}/v)$. Let now $\mathbf{y}$ and $\mathbf{z}$ be two arbitrary unit vectors orthogonal to each other and orthogonal to $\mathbf{v}$; we set

(2.6) $$P(\mathbf{y})^i{}_j = y^i y_j \ ; \qquad P(\mathbf{z})^i{}_j = z^i z_j$$

Of course we have

(2.7) $$P(\mathbf{y})^i{}_k P(\mathbf{y})^k{}_j = P(\mathbf{y})^i{}_j \ ; \qquad P(\mathbf{z})^i{}_k P(\mathbf{z})^k{}_j = P(\mathbf{z})^i{}_j \ ;$$
$$P(\mathbf{y})^i{}_k P(\mathbf{z})^k{}_j = P(\mathbf{z})^i{}_k P(\mathbf{y})^k{}_j = 0 \ ; \qquad \Omega^i{}_j = P(\mathbf{y})^i{}_j + P(\mathbf{z})^i{}_j \ ;$$
$$\Pi^i{}_j + P(\mathbf{y})^i{}_j + P(\mathbf{z})^i{}_j = \delta^i{}_j$$

In a manner similar to eqs (2.4), (2.5) we set

(2.8) $$a^i_y = P(\mathbf{y})^i{}_j a^j \quad \Leftrightarrow \quad \mathbf{a}_y = \mathbf{P(y)}\ \mathbf{a}$$

(2.9) $$a^i_z = P(\mathbf{z})^i{}_j a^j \quad \Leftrightarrow \quad \mathbf{a}_z = \mathbf{P(z)}\ \mathbf{a}$$

We are now in a position to state the following

THEOREM 2.1. *For every matrix $S^i{}_j$ we have*

(2.10) $$\begin{aligned} S^i{}_j = {} & b\frac{v^i}{v}\frac{v_j}{v} + b_2\frac{v^i}{v}y_j + b_3\frac{v^i}{v}z_j \\ & + d_2 y^i \frac{v_j}{v} + d\, y^i y_j + d_3 y^i z_j \\ & + e_2 z^i \frac{v_j}{v} + e_3 z^i y_j + e z^i z_j \end{aligned}$$

where $b, b_2, b_3, d_2, d, d_3, e_2, e_3, e$ are suitable scalar coefficients.

Of course every coefficient appearing in eq. (2.10) can be determined by the matrix $S^i{}_j$ itself; for instance

$$b_2 = \frac{v_i}{v} S^i{}_j y^j$$

We note that if we choose the unit vectors $\mathbf{v}/v$, $\mathbf{y}, \mathbf{z}$ as the canonical basis, then the projection operators specialize as

$$\text{(2.11)} \qquad \mathbf{\Pi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{P(y)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{P(z)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the matrix $\mathbf{S}$ writes

$$\text{(2.12)} \qquad \mathbf{S} = \begin{pmatrix} b & b_2 & b_3 \\ d_2 & d & d_3 \\ e_2 & e_3 & e \end{pmatrix}$$

We explicit remark that we shall use expressions (2.11) and (2.12) only when, in ref [3], a specific comparison with the results of Mansouri and Sexl [2] is carried out; otherwise we maintain all our results independent of the choice of the spatial coordinates.

3. Rotational invariance

Here we want to draw the consequences of assuming that the following requirement holds:

(R) the matrix $\mathbf{S}$ is invariant under rotation around the direction of $\mathbf{v}$

Let us make condition (R) mathematically operative. On denoting by $\mathbf{1}$ the identity matrix and by $\mathbf{R}^{\mathrm{T}}$ the transpose matrix of $\mathbf{R}$, the matrix $\mathbf{R}$ satisfies the orthogonality condition $\mathbf{RR}^{\mathrm{T}} = \mathbf{1}$. As is well known by the Euler theorem (see, e.g., ref [4]), the matrix $\mathbf{R}$ describes a rotation around $\mathbf{v}$ if and only if

$$\text{(3.1)} \qquad \mathbf{Rv} = \mathbf{v}$$

Under the action of $\mathbf{R}$ the two vectors $\mathbf{y}$ and $\mathbf{z}$ undergo a rotation of angle θ in the plane orthogonal to $\mathbf{v}$ which can be conveniently described in the following way

$$\text{(3.2a)} \qquad \mathbf{Ry} = \cos\theta\, \mathbf{y} + \sin\theta\, \mathbf{z}$$
$$\text{(3.2b)} \qquad \mathbf{Rz} = -\sin\theta\, \mathbf{y} + \cos\theta\, \mathbf{z}$$

Condition (R) now takes a precise mathematical form: for every rotation matrix $\mathbf{R}$ satisfying condition (3.1) we have

$$\text{(3.3)} \qquad \mathbf{S} = \mathbf{RSR}^{\mathrm{T}}$$

Consider now the rotation matrix $\mathbf{R}$ corresponding to a rotation of $\pi/2$ around $\mathbf{v}$; then conditions (3.1) and (3.2) become

$$\text{(3.4)} \qquad \mathbf{Rv} = \mathbf{v}\,, \quad \mathbf{Ry} = \mathbf{z}\,, \quad \mathbf{Rz} = -\mathbf{y}$$

Imposing condition (3.3) and using the relations (3.4), eq. (2.10) becomes

$$\text{(3.5)} \qquad \begin{aligned} S^i{}_j = R^i{}_p S^p{}_q R_j{}^q \; &= b\frac{v^i}{v}\frac{v_j}{v} + b_2\frac{v^i}{v}z_j - b_3\frac{v^i}{v}y_j \\ &+ d_2 z^i\frac{v_j}{v} + d\,z^i z_j - d_3 z^i y_j \\ &- e_2 y^i\frac{v_j}{v} - e_3 y^i z_j + e\,y^i y_j \end{aligned}$$

Comparing eq. (3.5) with eq. (2.10) yields

$$\text{(3.6)} \qquad b_2 = b_3 = 0\,, \quad d_2 = e_2 = 0\,, \quad d_3 = -e_3\,, \qquad d = e$$

Therefore eq. (2.10) becomes

$$\text{(3.7)} \qquad S^i{}_j = b\frac{v^i v_j}{v^2} + d(y^i y_j + z^i z_j) + d_3(y^i z_j - z^i y_j)$$

To proceed further, define the antisymmetric matrix

$$\text{(3.8)} \qquad M^i{}_j = y^i z_j - z^i y_j$$

and recall that, in view of eqs (2.1), (2.2), (2.6), (2.7), we have $\Pi^i{}_j = v^i v_j / v^2$ and $\Omega^i{}_j = y^i y_j + z^i z_j$. We state the following

THEOREM 3.1. *The matrices $\mathbf{\Pi}$, $\mathbf{\Omega}$ and $\mathbf{M}$ are invariant under rotation around the direction of $\mathbf{v}$, namely for every rotation matrix $\mathbf{R}$ satisfying eq. (3.1) the following relations*

$$\text{(3.9)} \qquad \mathbf{R\Pi R}^{\mathrm{T}} = \mathbf{\Pi}\,, \qquad \mathbf{R\Omega R}^{\mathrm{T}} = \mathbf{\Omega}\,, \qquad \mathbf{RMR}^{\mathrm{T}} = \mathbf{M}$$

hold true.

Proof. Indeed the procedure which led us to eq. (3.7) can be regarded as a proof of the theorem. Here, however, we prefer to prove the theorem by means of direct calculation. Precisely, in view of the orthogonality condition of $\mathbf{R}$ and of eq. (3.1) we have

$$R^i{}_p \Pi^p{}_q R_j{}^q = R^i{}_p v^p v_q R_j{}^q / v^2 = v^i v_j / v^2 = \Pi^i{}_j$$

$$R^i{}_p \Omega^p{}_q R_j{}^q = R^i{}_p (\delta^p{}_q - v^p v_q / v^2) R_j{}^q = R^i{}_p \delta^p{}_q R_j{}^q + R^i{}_p v^p v_q R_j{}^q / v^2 = \delta^i{}_j - v^i v_j / v^2 = \Omega^i{}_j$$

To prove the invariance of $\mathbf{M}$ we have to appeal to conditions (3.2). We get

$$R^i{}_p M^p{}_q R_j{}^q = R^i{}_p (y^p z_q - z^p y_q) R_j{}^q =$$

$$(\cos\theta\, y^i + \sin\theta\, z^i)(-\sin\theta\, y_j + \cos\theta\, z_j) - (-\sin\theta\, y^i + \cos\theta\, z^i)(\cos\theta\, y_j + \sin\theta\, z_j) = y^i z_j - z^i y_j = M^i{}_j$$

■

In view of eq. (3.7), we can state now the main result of this section, that is the following

THEOREM 3.2. *The matrix* $\mathbf{S}$ *is invariant under rotation around the direction of* $\mathbf{v}$ *if and only if it has the form*

$$\mathbf{S} = b\mathbf{\Pi} + d\mathbf{\Omega} + d_3\mathbf{M} \tag{3.10}$$

As a final remark, we note that the matrix $\mathbf{M}$ can be also written as

$$M^i{}_j = \varepsilon^i{}_{jp}\frac{v^p}{v} \tag{3.11}$$

where $\varepsilon^i{}_{jp}$ is the antisymmetric Levi-Civita tensor. This fact can be proved directly by multiplying both members of eq. (3.11) by y^i, z^i, y_j, z_j and verifying that, in view of eq. (3.8) we get the same result. It seems interesting to observe that all the projection operators appearing in eq. (3.10) can be written by using the vector $\mathbf{v}$ only.

4. Frame transformation

The results obtained so far concern a single frame. Things are slightly different when two frames, say A and F, are involved. On letting $\mathbf{v}$ be the velocity of F with respect to A, the coordinates X^i of F can be written in terms of the coordinates x^i of A according to the frame transformation

$$X^i = S^i{}_j(x^j - v^j t) \tag{4.1}$$

Introduce now two sets of unit orthogonal vectors in both frames: $\mathbf{v}/v, \mathbf{y}, \mathbf{z}$ in A and $\mathbf{V}/V, \mathbf{Y}, \mathbf{Z}$ in F where $\mathbf{V}$ is the velocity of A with respect to F. In view of the results of Sect. 2 each set of unit orthogonal vectors singles out the corresponding set of projectors. Also, for the transformation matrix $S^i{}_j$ can be proved a theorem analogous to Theorem 2.1, namely

THEOREM 4.1. *For the transformation matrix* $S^i{}_j$ *we have*

$$\begin{aligned} S^i{}_j = {} & b\frac{V^i}{V}\frac{v_j}{v} + b_2\frac{V^i}{V}y_j + b_3\frac{V^i}{V}z_j \\ & + d_2Y^i\frac{v_j}{v} + dY^i y_j + d_3Y^i z_j \\ & + e_2Z^i\frac{v_j}{v} + e_3Z^i y_j + eZ^i z_j \end{aligned} \tag{4.2}$$

where $b, b_2, b_3, d_2, d, d_3, e_2, e_3, e$ *are suitable scalar coefficients.*

It is interesting to remark that, when the relative velocities $\mathbf{v}$ and $\mathbf{V}$ are parallel, projecting parallel and orthogonal to $\mathbf{v}$ is frame independent: accordingly the two projectors can be written as

$$\mathbf{\Pi} = \mathbf{P}(\mathbf{v}/v) = \mathbf{P}(\mathbf{V}/V), \qquad \mathbf{\Omega} = \mathbf{1} - \mathbf{\Pi} \tag{4.3}$$

where $\mathbf{1}$ denote the identity operator.

5. An example: the Lorentz transformations

As an outstanding example we consider the celebrated Lorentz transformation between two inertial frames (see, e.g., ref. [5])

(5.1a) $$T = \gamma\left(t - \frac{\mathbf{v}\cdot\mathbf{x}}{c^2}\right)$$

(5.1b) $$\mathbf{X} = \mathbf{x} + \mathbf{v}\left[(\gamma-1)\frac{\mathbf{v}\cdot\mathbf{x}}{c^2} - \gamma t\right] = \mathbf{S}(\mathbf{x} - \mathbf{v}t)$$

where, as usual, $\gamma = 1/\sqrt{1 - v^2/c^2}$. Note that the matrix $\mathbf{S}$ has the form

(5.2) $$\mathbf{S} = \gamma\mathbf{\Pi} + \mathbf{\Omega}$$

and hence it satisfies the invariance condition (R).

References

[1] F. Bampi, C. Zordan, *Ann. Phys.* 190, 428-444 (1989).
[2] R. Mansouri, R.U. Sexl, *Gen. Rel. Grav.* 8, 497-513 (1977).
[3] F. Bampi, C. Zordan, *High velocity frame transformations. II: A new approach to Mansouri and Sexl theory*, Scientific and Technical Notes - Univ. Genova, to be published
[4] H. Goldstein, *Classical mechanics*, Addison-Wesley, Reading (Mass.) 1980.
[5] C. Möller, *The theory of Relativity*, Oxford University Press, Oxford 1972.