Hamiltonian cosmology and gravitational solitons: A variational outlook

F. Bampi and A. Morro
Istituto di Matematica dell'Università, Via L.B. Alberti 4, 16132 Genova, Italy
(Received 1 July 1982)

A general procedure is applied for finding the possible variational principle of a given nonlinear system of partial differential equations. In particular, attention is focused on Hamiltonian homogeneous cosmologies and gravitational solitons. In the first case, it is found that a variational principle exists if and only if the model is of class A, thus complementing previous results of MacCallum and Taub. In the second case, it is shown that the pertinent system arises from a variational principle; furthermore, the explicit form of the Lagrangian is given. Owing to the structure of the Lagrangian an outstanding first integral is determined.

I. INTRODUCTION

There are a number of features weighing in favor of variational formulations versus their local counterparts: unification of diverse fields, methods for approximating or finding the solution, connection between symmetries and conservation laws. Moreover, it is a general belief that a Lagrangian is more fundamental than the resulting Euler-Lagrange equations; this happens, for example, in Feynman's path integral of quantum-mechanical systems. It is then hardly surprising that the search for a Lagrangian corresponding to a given system of equations (inverse problem) has received so wide an attention.[1] In particular, Vainberg's theorem specifies the conditions under which a system of equations admits a variational derivation. Furthermore, if such conditions are met, Vainberg's theorem yields an operative procedure to determine the desired functional. How this theorem is effective has been shown extensively in a previous paper[2] concerning various topics pertaining to continuum (nonrelativistic) physics.

It is the aim of this paper to examine two outstanding topics of relativity from the inverse-problem standpoint by appealing again to Vainberg's theorem. First, Hamiltonian cosmology has been widely used for explicating complicated motions of the universe; one of the most striking results of the Hamiltonian cosmology is that the dynamics of some types of universe is shown to be equivalent to that of very simple mechanical systems.[3] However, in connection with homogeneous cosmologies, troubles arise to such an extent that the death of Hamiltonian cosmology has been claimed.[4] On the basis of this motivation, here we revisit homogeneous cosmologies from the inverse-problem viewpoint; our analysis will confirm decisively known results on the subject. Second, the search for soliton solutions of Einstein's equations is paid more and more attention mainly because the underlying technique provides a powerful tool for finding broad classes of solutions with a large degree of freedom and varying physical content. Further, up to now no equation having a soliton solution has been shown not to be the local counterpart of some variational problem. This suggests that we look at the Einstein soliton equation[5] with the purpose of arriving at its possible variational account; in fact, we will succeed in determining the explicit expression for the Lagrangian.

The relativistic fluid as well is a challenging subject for a variational investigation. As to the Eulerian description, Vainberg's theorem allows us to conclude that Schutz's paper[6] is a satisfactory answer to the problem. As to the Lagrangian description, instead, new difficulties arise which make its study beyond the scope of this paper.

II. POTENTIALNESS OF A SYSTEM OF DIFFERENTIAL EQUATIONS

For later reference, in this section we gather the essentials of Vainberg's theorem and its consequences relevant to the case under consideration; the reader desiring a detailed analysis of the subject is referred to Refs. 1 and 2 and references therein.

Let X be a Hilbert space of functions from a domain $\mathscr{D} \subset \mathsf{R}^n$ into an m-dimensional vector space V and let a centered dot denote the inner product on V. According to Vainberg's theorem, a necessary and sufficient condition for a (nonlinear) operator $\mathsf{N}: X \rightarrow X$ to be the gradient of a functional (potentialness) in a ball $B \subset X$ is that the Gâteaux differential $DN(\cdot \mid \cdot)$ satisfies

$$\int_{\mathscr{D}} DN(u \mid h)\cdot k\, dv = \int_{\mathscr{D}} DN(u \mid k)\cdot h\, dv \quad (2.1)$$

for every $h,k \in X$ and every $u \in B$. Moreover, if (2.1) is satisfied, then the Lagrangian (density) $L(u)$ of N is given by

$$L(u) = (u - u_0)\cdot \int_0^1 N(u_0 + \lambda(u - u_0))d\lambda \ . \quad (2.2)$$

It is worth writing down a necessary and sufficient condition for (2.1) to be true. Specifically, let N be expressed by the system of second-order differential equations

$$f_A(u^B, u^B{}_{,p}, u^B{}_{,pq}, x^p) = 0 \quad (2.3)$$

in the unknown functions u^B on the n variables x^p, $p = 1, \ldots, n$. A comma denotes partial differentiation while A and B run over a suitable set of indices; for example, $A = 1, \ldots, m$ or $A = (i,j)$, $i,j = 1, \ldots, m$. The system (2.3) meets the potentialness condition (2.1) if and only if

$$\left[\frac{\partial f_A}{\partial u^B{}_{,pq}} - \frac{\partial f_B}{\partial u^A{}_{,pq}} \right] k^A h^B = 0 \ ,$$

$$\left[\frac{\partial f_A}{\partial u^B{}_{,p}} + \frac{\partial f_B}{\partial u^A{}_{,p}} - 2\left[\frac{\partial f_B}{\partial u^A{}_{,pq}} \right]_{,q} \right] k^A h^B = 0 \ , \quad (2.4)$$

$$\left[\frac{\partial f_A}{\partial u^B} - \frac{\partial f_B}{\partial u^A} + \left[\frac{\partial f_B}{\partial u^A{}_{,p}} \right]_{,p} + \left[\frac{\partial f_B}{\partial u^A{}_{,pq}} \right]_{,pq} \right] k^A h^B = 0 \ ,$$

for every set of admissible quantities k^A, h^B. The following sections deliver a straightforward application of (2.4) to homogeneous cosmologies and gravitational solitons.

According to Vainberg's theorem the indices A,B need to be tensorial in character. If, however, the tensor character is in order, then we have preliminarily to ascertain that f_A and $\partial/\partial u^A$ are endowed with the same covariance properties. The form of (2.4) complies with this prescription.

III. HAMILTONIAN HOMOGENEOUS COSMOLOGIES

The study of cosmological models in Hamiltonian form is based on the Arnowitt-Deser-Misner (ADM) method which describes the dynamics of geometry through a 3+1 decomposition of the Einstein's action function and the corresponding field equations.[3] As was first noted by Hawking,[7] simply inserting a homogeneous metric into the ADM action functional does not lead always to the correct Einstein equations. In particular, the lack of a variational formulation for the Bianchi-type universe of class B was pointed out by MacCallum and Taub.[8] Unfortunately, as the literature shows,[7–10] the derivation of field equations is usually dealt with in a manner which suffers from serious drawbacks concerning the behavior of the variations of the independent variables at the boundary. As an example of such drawbacks, we mention that the action itself is infinity merely because of the spatial contribution.[10] To our mind, the best way of avoiding these conceptual difficulties and, meanwhile, of arriving at definitive conclusions is to look at the correct ADM equations for homogeneous metrics and to test their potentialness through the procedure outlined in Sec. II.

Here we adopt ADM notations. As usual in cosmology, we let $N_i = 0$, $N = 1$. Accordingly, the ADM version of Einstein's equations reads

$$\begin{aligned} \Gamma^{ab} \equiv \pi^{ab}{}_{,t} &+ g^{1/2}({}^3R^{ab} - \tfrac{1}{2}\,{}^3R g^{ab}) \\ &- \tfrac{1}{2} g^{-1/2} g^{ab}(\pi^{pq}\pi_{pq} - \tfrac{1}{2}\pi^2) \\ &+ 2g^{-1/2}(\pi^{ap}\pi_p{}^b - \tfrac{1}{2}\pi\pi^{ab}) = 0 \ , \end{aligned} \quad (3.1)$$

$$P_{ab} \equiv -g_{ab,t} + 2g^{-1/2}(\pi_{ab} - \tfrac{1}{2}\pi g_{ab}) = 0 \ . \quad (3.2)$$

In the case of spatially homogeneous models (characterized by the existence of a three-parameter isometry group which is transitive on a family of spacelike hypersurfaces) it is convenient to introduce, on every surface of homogeneity $t =$const, an anholonomic frame invariant under the action of the isometry group. In so doing the spatial metric g_{ab} and the momentum density π^{ab} turn out to be functions on the time t only. In particular, the three-dimensional Ricci tensor takes the form

$$\begin{aligned} {}^3R^{ab} = &-\tfrac{1}{2} C^k_{ms} C^m_{kr} g^{sa} g^{rb} \\ &- \tfrac{1}{2} C^m_{rs} C^n_{kj} g_{mn} g^{rk} g^{sa} g^{jb} \\ &+ \tfrac{1}{4} C^a_{sh} C^b_{kr} g^{sk} g^{hr} \\ &- a_r C^b_{km} g^{rk} g^{ma} - a_r C^a_{kn} g^{rk} g^{nb} \ , \end{aligned} \quad (3.3)$$

C^b_{kj} being the structure constants of the group specifying the Bianchi type and $a_r = \tfrac{1}{2} C^b_{rb}$. Mathematically, the problem now is to investigate the system of equations (3.1) and (3.2)—together with the homogeneity assumption (3.3)—in the unknown functions g_{ab}, π^{ab} and to determine the conditions under which it admits a variational principle.

Look at (2.4) in connection with the system (3.1) and (3.2). In this context the requirement that

$$\left[\frac{\partial \Gamma^{ab}}{g_{pq}} - \frac{\partial \Gamma^{pq}}{g_{ab}} \right] h_{ab} k_{pq} = 0 \ , \tag{3.4}$$

for arbitrary symmetric tensors h_{ab}, k_{pq}, is the only one which is not identically satisfied. Difficulties occur, in fact, in connection with the quantity

$$ {}^3G^{ab} = {}^3R^{ab} - \tfrac{1}{2}\, {}^3R g^{ab} \ .$$

To see this, observe first that (3.4) reduces to

$$\left[g^{pq} \left[{}^3R^{ab} + \tfrac{1}{2} g_{cd} \frac{\partial\, {}^3R^{cd}}{\partial g_{ab}} \right] - g^{ab} \left[{}^3R^{pq} + \tfrac{1}{2} g_{cd} \frac{\partial\, {}^3R^{cd}}{\partial g_{pq}} \right] \right.$$

$$\left. + \frac{\partial\, {}^3R^{ab}}{\partial g_{pq}} - \frac{\partial\, {}^3R^{pq}}{\partial g_{ab}} \right] h_{ab} k_{pq} = 0 \ . \tag{3.5}$$

On the other hand, in view of (3.3), tedious calculations yield

$$ {}^3R^{pq} + \tfrac{1}{2} g_{cd} \frac{\partial\, {}^3R^{cd}}{\partial g^{pq}} = 2a^p a^q - a_r C^q_{sm} g^{rs} g^{mp} - a_r C^p_{sm} g^{rs} g^{mq} \ , \tag{3.6}$$

$$\frac{\partial\, {}^3R^{ab}}{\partial g^{pq}} - \frac{\partial\, {}^3R^{pq}}{\partial g_{ab}} = 2[C^{(pq)(a}a^{b)} - C^{(ab)(p}a^{q)} + g^{s(p}g^{q)(a}C^{b)}{}_{rs}a^r - g^{s(a}g^{b)(p}C^{q)}{}_{rs}a^r] \ , \tag{3.7}$$

where parentheses denote symmetrization. Substitution of these results into (3.5) provides the sought conditions for the system (3.1) and (3.2) to admit a variational principle.

To examine these conditions in more detail we put $h_{ab} = g_{ab}$; the arbitrariness of k_{pq} allows us to find that

$$2a^p a^q - a^r g^{s(p} C^{q)}{}_{rs} = 0 \ . \tag{3.8}$$

As shown by MacCallum and Taub,[8] (3.8) is equivalent to

$$a^p = 0 \tag{3.9}$$

which, according to (3.6), (3.7), makes (3.5) identically satisfied.

In conclusion, Vainberg's theorem implies that the dynamical equations (3.1) and (3.2) for spatially homogeneous cosmologies may be described through a variational principle if and only if the vector a^p vanishes. In other words, a variational formulation is possible if and only if the space is of class A.

We end this section with two comments. First, condition (3.9) has already been given in Refs. 8–10; these authors simply show that, on imposing spatial homogeneity on the ADM Lagrangian, the correct equations (3.1) and (3.2) are arrived at if and only if (3.9) is true. Nevertheless, this result does not preclude finding any variational principle for the system (3.1) and (3.2), together with the assumption (3.3). Here, instead, we have proved such a preclusion to hold in that the system (3.1) and (3.2) admits a variational formulation if and only if (3.9) is satisfied.

Second, we point out an unusual application of Vainberg's theorem. On account of (2.2), calculate first the function that coincides with the Lagrangian when the system under consideration is potential and then derive the corresponding Euler-Lagrange equations. Accordingly, the system is potential if and only if such equations are just the original equations. In our case, this procedure gives the ADM Lagrangian which, as we know, leads to incorrect equations for spaces of class B. This observation confirms our results and corroborates those in Refs. 8–10.

IV. CYLINDRICALLY AND AXIALLY SYMMETRIC GRAVITATIONAL SOLITONS

In investigating Einstein's equations, soliton solutions have received increasing attention in recent years. The interest in the variational formulation of equations admitting soliton solutions is motivated on a threefold basis. First, there are equations (e.g., the Korteweg-de Vries equation) which are derived through an approximate procedure; the derived equation need not be conservative. The existence of a variational formulation for the derived equation allows the equation itself to be considered conservative or lossless in the conventional sense of the term.[11] Second, the availability of a functional leads directly to the construction of constants of motion or conserved quantities by appealing to the invariance properties of the functional itself. Of course, these considerations apply to any type of equations irrespective of the classical or the relativistic contexts being concerned while the following third motivation is typical of relativity. The field equations may be written by taking into account suitable symmetry requirements; as shown in the previous section, these symmetry requirements may determine the failure of the variational formulation. That is why the variational structure of an equation must be ascertained in every case; here we are dealing with the gravitational soliton equation.

Following Ref. 5, the metric is supposed to be

$$ds^2=f(z,t)(dz^2-e\,dt^2)+g_{ab}(z,t)dx^a dx^b,\quad a,b=1,2\ ,$$

e being equal to -1 or $+1$ according as axially symmetric stationary or cylindrically symmetric fields are considered. Then, letting

$$\alpha^2=e\det(g_{ab})\ , \tag{4.1}$$

Einstein's vacuum field equations determine the 2×2 matrix g_{ab} through

$$H_a{}^b\equiv(\alpha g_{ac,z}g^{cb})_{,z}-e(\alpha g_{ac,t}g^{cb})_{,t}=0\ , \tag{4.2}$$

while the function f may be evaluated by quadratures once a solution to (4.1) is found. So, the trace of (4.2) yields the well-known result[5]

$$\alpha_{,zz}-e\alpha_{,tt}=0\ . \tag{4.3}$$

It is a remarkable feature of (4.2) that we may disregard the condition (4.1) while performing calculations, provided we renormalize the final result as[5,12]

$$g_{ab}\to\alpha[e\det(g_{rs})]^{-1/2}g_{ab}\ . \tag{4.4}$$

Accordingly, we assume α to be an assigned solution to (4.3) and then we apply (4.4) to get the correct physical result.

It is a routine matter to ascertain that the potentialness condition (2.4) for the system $g^{ac}H_c{}^b=0$, equivalent to (4.2), is satisfied. Then we move on to determine the corresponding Lagrangian $L(g_{ab})$. According to (2.2) we have

$$L(g_{rs})=(g_{ab}-\hat{g}_{ab})\int_0^1\tilde{g}^{ac}[(\alpha\tilde{g}_{cd,z}\tilde{g}^{db})_{,z}-e(\alpha\tilde{g}_{cd,t}\tilde{g}^{db})_{,t}]d\lambda\ , \tag{4.5}$$

where $\hat{g}_{ab}$ is an arbitrary fixed value of g_{ab} while $\tilde{g}_{ab}=\hat{g}_{ab}+\lambda(g_{ab}-\hat{g}_{ab})$ and $\tilde{g}^{ab}$ is the inverse matrix of $\tilde{g}_{ab}$. Without any loss of generality we choose the metric $\hat{g}_{ab}$ to be independent of z,t.

The evaluation of the integral in (4.5) proceeds by way of intermediate stages; we will show some details because they are typical when determining Lagrangians corresponding to nonlinear differential operators. To begin with, observe that

$$\frac{\partial\tilde{g}^{cd}}{\partial\lambda}=-\tilde{g}^{cr}\tilde{g}^{ds}(g_{rs}-\hat{g}_{rs})\ . \tag{4.6}$$

Now, denote by L_z the part of L involving the derivatives with respect to z. Some rearrangement and use of (4.6) yields

$$L_z=I_1-I_2-I_3\ , \tag{4.7}$$

where

$$I_1=\alpha\int_0^1\frac{\partial\tilde{g}^{cd}}{\partial\lambda}\tilde{g}_{cm,z}\tilde{g}_{dn,z}\tilde{g}^{mn}d\lambda\ ,$$

$$I_2=\alpha\int_0^1\frac{\partial\tilde{g}^{cd}}{\partial\lambda}\tilde{g}_{cd,zz}d\lambda\ ,$$

$$I_3=\alpha_{,z}\int_0^1\frac{\partial\tilde{g}^{cd}}{\partial\lambda}\tilde{g}_{cd,z}d\lambda\ .$$

Since $\tilde{g}_{ab,z}=\lambda g_{ab,z}$, integration by parts gives

$$I_1=\tfrac{1}{2}\alpha g^{cd}g^{mn}g_{cm,z}g_{dn,z}+\alpha g_{cd,z}\int_0^1\tilde{g}^{cd}{}_{,z}d\lambda\ ,$$

$$I_2=\alpha g^{cd}g^{mn}g_{cm,z}g_{dn,z}+\alpha g_{cd,z}\int_0^1\tilde{g}^{cd}{}_{,z}d\lambda-\alpha_{,z}g^{cd}g_{cd,z}+\alpha_{,z}g_{cd,z}\int_0^1\tilde{g}^{cd}d\lambda\ ,$$

$$I_3=\alpha_{,z}g^{cd}g_{cd,z}-\alpha_{,z}g_{cd,z}\int_0^1\tilde{g}^{cd}d\lambda\ ;$$

in writing the expression for I_2 a derivative with respect to z has been omitted because it affects the action functional only through (inessential) boundary terms. Substitution in (4.7) produces the expression for L_z.

By simply replacing z with t we get the expression for the remaining part L_t of L. Collecting these results we arrive at the sought Lagrangian

$$L(g_{rs})=\tfrac{1}{2}\alpha g^{cd}g^{mn}(g_{cm,z}g_{dn,z}-eg_{cm,t}g_{dn,t})\ . \tag{4.8}$$

Two remarks are in order. First, symmetries of a Lagrangian imply the existence of conservation laws. Now, in connection with the Lagrangian (4.8) possible symmetries are strictly related to the structure of $\alpha(z,t)$. In fact, as shown, e.g., in Refs. 12 and 13, without any loss of generality we may choose the coordinate z in such a way that $\alpha(z,t)=z$. This in turn yields the conservation law

$$\mathscr{H}\equiv\tfrac{1}{2}zg^{cd}g^{mn}(g_{cm,z}g_{dn,z}+eg_{cm,t}g_{dn,t})=\text{const}\ . \tag{4.9}$$

The formidable system (4.2)—and the equivalent system in Weyl canonical coordinates (see Ref. 14, p. 223)—bears evidence of the importance of the first integral (4.9). Second, the value of the result (4.8) is strengthened by the fact that the Hilbert Lagrangian $L_H=(-{}^4g)^{1/2}\,{}^4R$ does not lead to the correct Einstein equations (4.2). Specifically, in the case under consideration it follows that[13]

$$L_H = e\alpha(\ln f)_{,tt} - \alpha(\ln f)_{,zz} + e\alpha g^{ab}g_{ab,tt} - \alpha g^{ab}g_{ab,zz} + \tfrac{1}{4}e\alpha(g^{ab}g_{ab,t})^2 - \tfrac{1}{4}\alpha(g^{ab}g_{ab,z})^2 + \tfrac{3}{2}L(g_{rs})$$

whence the desired equations for f are not achieved. Nevertheless, even if, by analogy with an attempt undertaken in Hamiltonian cosmology,[9] we confine attention to the reduced Lagrangian L_1 obtained from L_H by setting $f=1$, the corresponding equations are

$$\tfrac{1}{2}g^{ac}H_c{}^b + \tfrac{1}{2}g^{ab}L_1 = 0 .$$

Since, in general, $L_1 \neq 0$ on the solutions of (4.2), the inconsistency with (4.2) is evident. Thus we have found a further example where imposing symmetries on the Hilbert Lagrangian makes the Euler-Lagrange equations inequivalent to the correct Einstein equations.

We note in passing that if the specification (4.1) is taken into account, then the potentialness conditions are not met. On the other hand, there are no counterexamples to the conjecture that equations admitting soliton solutions arise from variational principles. Accordingly, the previous observation complies with the feature that α being an assigned function on z,t is crucial for the application of the inverse-scattering-problem technique leading to soliton solutions.[5]

[1]R. M. Santilli, *Foundations of Theoretical Mechanics I: The Inverse Problem in Newtonian Mechanics* (Springer, Berlin, 1978).

[2]F. Bampi and A. Morro, J. Math. Phys. (to be published).

[3]M. Ryan, *Hamiltonian Cosmology* (Lecture Notes in Physics 13) (Springer, Berlin, 1972).

[4]M. Ryan, J. Math. Phys. 15, 812 (1974).

[5]V. A. Belinskii and V. E. Zakharov, Zh. Eksp. Teor. Fiz. 75, 1955 (1978) [Sov. Phys. JETP 48, 985 (1978)].

[6]B. F. Schutz, Jr., Phys. Rev. D 2, 2762 (1970).

[7]S. Hawking, Mon. Not. R. Astron. Soc. 142, 129 (1969).

[8]M. A. H. MacCallum and A. H. Taub, Commun. Math. Phys. 25, 173 (1972).

[9]G. E. Sneddon, J. Phys. A 9, 229 (1976).

[10]R. T. Jantzen, Nuovo Cimento B55, 161 (1980).

[11]A. C. Scott, F. Y. F. Chu, and D. W. McLaughlin, Proc. IEEE 61, 1443 (1973).

[12]F. Bampi, Gen. Relativ. Gravit. 14, 429 (1982).

[13]V. A. Belinskii and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 57, 2163 (1969) [Sov. Phys. JETP 30, 1174 (1970].

[14]D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein Field Equations* (VEB Deutscher Verlag der Wissenschaften, Berlin, 1980).