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Effects of Viscosity on Water Waves.

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Summary. — The paper deals with the behaviour of a viscous liquid whose flow preserves the structure of the material columns. A balance law for energy is established which accounts for Navier-Stokes dissipation; on appealing to the invariance of such a law under rigid motions, balance equations for mass and linear momentum are derived. It is an outstanding consequence of the theory that the simultaneous occurrence of viscosity and inertia terms makes long gravity waves be governed by the combined Korteweg-de Vries and Burgers equation.

1. – Introduction.

Standard accounts of long gravity waves of finite amplitude are based upon the simplifying assumption that the viscosity is totally negligible (1). Despite this drastic simplification, serious difficulties still occur because of the presence of nonlinear inertia terms and of the nonlinear boundary condition over the unknown free surface. That is why the literature bears evidence of several approximate methods making the problem handier (2).

Lately, in order to set up more realistic models for gravity waves in fluids, viscosity has been receiving a great deal of attention. Some approaches describe viscosity by having recourse to a diffusion operator (3,4). Others describe vis-

(1) J. J. STOKER: *Water Waves* (New York, N. Y., 1957).
(2) See, *e.g.*, (1) and J. HAMILTON: *J. Fluid Mech.*, **83**, 289 (1977).
(3) E. OTT and R. N. SUDAN: *Phys. Fluids*, **13**, 1432 (1970).
(4) T. KAKUTANI and K. MATSUUCHI: *J. Phys. Soc. Jpn.*, **39**, 237 (1975); J. W. MILES: *Phys. Fluids*, **19**, 1063 (1976).

cosity through Navier-Stokes' law, but make use of linear approximations. Specifically, linearized boundary conditions were adopted both in dealing with long waves in shallow water ([5]) and in investigating small-amplitude standing surface waves in infinitely deep liquids ([6]).

In spite of the wide interest in the subject, a rigorous theory of gravity waves based on Navier-Stokes' equations is still lacking. A first contribution towards a better understanding of the role of viscosity in gravity waves is given in a paper of Mei ([7]), in which the diffusion length is assumed to be not too small as compared with the fluid depth; in this way he was able to retain nonlinear effects. It is just the aim of this paper to provide a new contribution by establishing a scheme which describes viscosity via Navier-Stokes' law and, meanwhile, accounts fully for the nonlinear boundary conditions. Precisely, basing upon the scheme elaborated by GREEN and NAGHDI ([8]) in connection with inviscid fluids, here we develop a model for gravity waves in viscous fluids under the only approximation that the fluid particles which, at some initial time, belong to a vertical (material) column will continue to belong to the same vertical column; ultimately this condition turns out to be similar to but more accurate than the shallow-water approximation ([9]).

Often, in studying wave propagation problems governed by nonlinear equations, one considers only the lowest approximation, namely the linearized counterpart. An approximate procedure, allowing also for nonlinear terms, may be performed by having recourse to stretching transformations. This second procedure, which has been applied for investigating the structure of shocks ([10]), is now widely used because of its great flexibility for describing dispersive and dissipative waves ([11]). For example, in connection with water waves, we mention an application of stretching transformations by means of which Green and Naghdi's inviscid model is shown to lead directly to the Korteweg-de Vries equation ([12,13]). Here we apply again this procedure so as to examine the effects of viscosity on the behaviour of nonlinear long waves.

Briefly, the plan of the paper is as follows. In sect. **2**, starting from the basic assumption of Green and Naghdi's column model, we derive the balance equations for mass and linear momentum from the energy equation of viscous

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([5]) M. I. G. BLOOR: *Phys. Fluids*, **13**, 1435 (1970).
([6]) M. YANOWITCH: *J. Fluid Mech.*, **29**, 209 (1977); A. PROSPERETTI: *Phys. Fluids*, **19**, 195 (1976).
([7]) C. C. MEI: *J. Math. Phys.* (*N. Y.*), **45**, 266 (1966).
([8]) A. E. GREEN and P. M. NAGHDI: *J. Fluid Mech.*, **78**, 237 (1976).
([9]) F. BAMPI and A. MORRO: *Nuovo Cimento C*, **1**, 377 (1978).
([10]) R. E. MEYER: *Structure of collisionless shocks*, in *Nonlinear Waves*, edited by S. LEIBOVICH and A. R. SEEBASS (London, 1977).
([11]) C. H. SU and C. S. GARDNER: *J. Math. Phys.* (*N. Y.*), **10**, 536 (1969).
([12]) F. BAMPI and A. MORRO: *Lett. Nuovo Cimento*, **26**, 61 (1979).
([13]) F. BAMPI and A. MORRO: *Nuovo Cimento C*, **2**, 352 (1979).

fluids via the invariance under superposed rigid-body motions. Then, as a particular case, in sect. **3** we obtain the shallow-water model for viscous fluids. Finally, in sect. **4** we are concerned with an outstanding application of the new theory. Precisely, after having ascertained that Su and Gardner's approach ([11]) does not apply to our equations, we analyse the consequences of a suitable stretching transformation. It is a noteworthy consequence that long gravity waves turn out to be governed by the combined Korteweg-de Vries and Burgers equation; this provides a new hydrodynamic motivation ([7,14]) for mathematical investigations of such a model equation ([15,16]).

2. – Column model.

Let x, y be horizontal space co-ordinates, z the vertical space co-ordinate and $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the associated orthonormal basis. Henceforth we consider an incompressible viscous fluid with constant mass density ϱ moving between the uneven bottom

$$\boldsymbol{x} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 - h(x, y)\,\boldsymbol{e}_3$$

and the free surface

$$\boldsymbol{x} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + \eta(x, y, t)\,\boldsymbol{e}_3\,,$$

where t is the time. The vertical co-ordinate is chosen is such a way that at equilibrium the free surface is $\eta(x, y, t) = 0$. The pressure at the free surface is the atmospheric pressure p_a, while the pressure P at the bottom is a function of x, y, t. As usual, a superposed dot denotes the material time derivative, while g is the gravity acceleration.

Roughly speaking, the column model ([8,9]) relies on the hypothesis that the elementary constituents of the fluid are infinitesimal vertical columns rather than the usual fluid particles. This is made precise by saying that the position of a particle of the fluid is expressed as

$$\boldsymbol{x} = \boldsymbol{r} + (\psi + Z\varphi)\,\boldsymbol{e}_3\,, \tag{2.1}$$

where $\boldsymbol{r} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2$, $\psi = (\eta - h)/2$, $\varphi = \eta + h$, $Z \in [-\frac{1}{2}, \frac{1}{2}]$; the free surface corresponds to $Z = \frac{1}{2}$, the bottom to $Z = -\frac{1}{2}$. In view of (2.1) the velocity $\boldsymbol{V} = \dot{\boldsymbol{x}}$ may be written as

$$\boldsymbol{V} = \boldsymbol{v} + (\lambda + Zw)\,\boldsymbol{e}_3\,, \tag{2.2}$$

([14]) R. S. Johnson: *J. Fluid Mech.*, **42**, 49 (1970).
([15]) A. Jeffrey and T. Kakutani: *SIAM Rev.*, **14**, 582 (1972).
([16]) D. J. Kaup: *Physica D (The Hague)*, **1**, 391 (1980).

where $\boldsymbol{v} = \dot{\boldsymbol{r}}$ is the horizontal component of the velocity, while $\lambda = \dot{\psi}$, $w = \dot{\varphi}$. As to the physical meaning of λ and w, we note that $\lambda = \lambda(x, y, t)$ is the vertical velocity of the centre of mass of the fluid column around (x, y), while Zw is the vertical velocity of the particles of the column, at $z = \psi + Z\varphi$, relative to the centre of mass.

It should be mentioned that, in conjunction with the velocity field (2.2), the horizontal velocity $\boldsymbol{v}$ at the bottom is different from zero unlike the usual assumption that $\boldsymbol{v} = \boldsymbol{0}$ at the bottom. Physically this makes the column model appropriate if the effects of viscous adherence are confined to the boundary layer in which the fluid meets the bottom.

To go further, it is convenient to look at an arbitrary fluid column occupying a time-dependent region $\mathscr{P}^*$ bounded by a closed cylinder $\partial\mathscr{P}^*_n$ whose unit outwards normal is denoted by $\boldsymbol{n}$ (see fig. 1). Moreover, denote by $\mathscr{P}$ the part of the surface $z = \psi(x, y, t)$ belonging to $\mathscr{P}^*$. These notations allow us to ex-

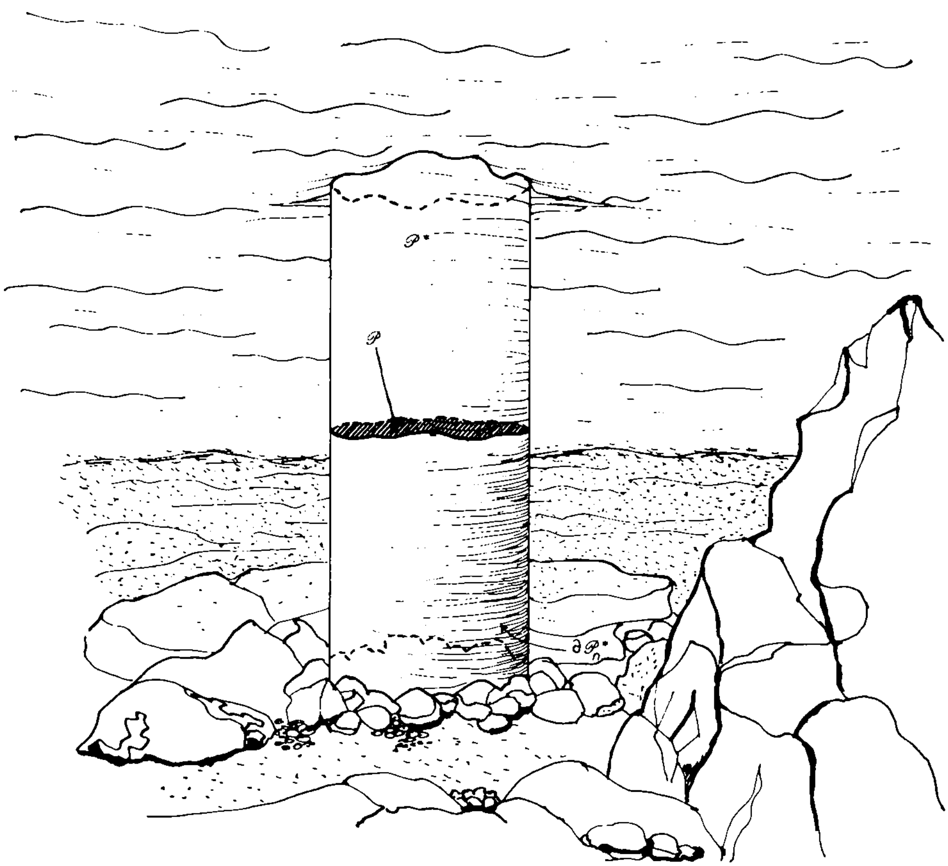

Fig. 1. – A typical fluid column: $\mathscr{P}^*$ is the column, $\partial\mathscr{P}^*_n$ the cylindrical boundary, $\mathscr{P}$ the intersection of the column with the surface $z = \psi(x, y, t)$.

press the energy balance as the natural generalization of the one corresponding to the inviscid approximation [8,9]. Letting the stress tensor $\boldsymbol{T}$ be given by the usual Navier-Stokes' law

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\mu\boldsymbol{D}\,, \qquad \operatorname{tr}\boldsymbol{D} = 0$$

for incompressible fluids, we have

$$\text{(2.3)} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*}\varrho\left(\frac{1}{2}V^2 + gz + e\right)\mathrm{d}V = \int_{\mathscr{P}^*}\varrho r\,\mathrm{d}V + \int_{\partial\mathscr{P}^*}[(-p\boldsymbol{n} + 2\mu\boldsymbol{D}\boldsymbol{n})\cdot\boldsymbol{V} - \boldsymbol{q}\cdot\boldsymbol{n}]\,\mathrm{d}a\,,$$

where e is the internal energy, r the rate of supply of external heat, $\boldsymbol{q}$ the heat flux vector.

Of course, the balance law (2.3) holds for arbitrary regions $\mathscr{P}^*$, which need not be column shaped. Moreover, eq. (2.3) must be invariant under superposed rigid-body motions, which means that the change of frame corresponding to the transformation $\boldsymbol{V}\to\boldsymbol{V}+\boldsymbol{U}$ must leave eq. (2.3) unaltered. Accordingly, the arbitrariness of the constant vector $\boldsymbol{U}$ allows us to derive the usual forms of the balance of the linear momentum as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*}(\varrho\boldsymbol{V} + g\boldsymbol{e}_3 t)\,\mathrm{d}V = \int_{\partial\mathscr{P}^*}(-p\boldsymbol{n} + 2\mu\boldsymbol{D}\boldsymbol{n})\,\mathrm{d}a$$

and, hence, of the balance of internal energy as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*}\varrho e\,\mathrm{d}V = \int_{\mathscr{P}^*}(2\mu\boldsymbol{D}\cdot\boldsymbol{D} + \varrho r)\,\mathrm{d}V - \int_{\partial\mathscr{P}^*}\boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}a\,.$$

As a consequence, the balance equation (2.3) simplifies to

$$\text{(2.4)} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*}\varrho\left(\frac{1}{2}V^2 + gz\right)\mathrm{d}V = \int_{\mathscr{P}^*}2\mu\boldsymbol{V}\cdot(\operatorname{div}\boldsymbol{D})\,\mathrm{d}V - \int_{\partial\mathscr{P}^*}p\boldsymbol{n}\,\mathrm{d}a\,.$$

Now we move on to exploit the balance equation (2.4) within the framework of the column scheme. First we observe that, in connection with the velocity field (2.2), the stretching tensor $\boldsymbol{D}$ takes the form

$$\boldsymbol{D} = \left(\begin{array}{c|c} \boldsymbol{d} & \frac{1}{2}\nabla(\lambda + Zw) \\ \hline \frac{1}{2}\nabla(\lambda + Zw) & -\nabla\cdot\boldsymbol{v}\end{array}\right),$$

where $\nabla = \boldsymbol{e}_1(\partial/\partial x) + \boldsymbol{e}_2(\partial/\partial y)$ and $\boldsymbol{d} = \operatorname{sym}(\nabla\boldsymbol{v})$. Thus the powers at the free surface σ_f and at the bottom σ_b are

$$\sigma_f = \int_{\mathscr{P}} p_a[\boldsymbol{v}\cdot\nabla\eta - (\lambda + \tfrac{1}{2}w)]\,\mathrm{d}x\,\mathrm{d}y\,, \qquad \sigma_b = \int_{\mathscr{P}} P[\boldsymbol{v}\cdot\nabla h - (\lambda - \tfrac{1}{2}w)]\,\mathrm{d}x\,\mathrm{d}y\,,$$

while the power σ_c at the cylindrical surface $\partial\mathscr{P}_n^*$ is expressed as

$$\sigma_c = -\,\boldsymbol{e}_3\cdot\int_{\partial\mathscr{P}} \Pi\boldsymbol{v}\times\mathrm{d}\boldsymbol{r}\,,$$

where

$$\Pi = \int_{-h}^{\eta} p(z)\,\mathrm{d}z\,.$$

As a result, the balance of energy for the column under consideration takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \frac{1}{2}\,\varrho\varphi\left(v^2 + \lambda^2 + \frac{1}{12}\,w^2 + 2g\psi\right)\mathrm{d}x\,\mathrm{d}y = \tag{2.5}$$

$$= \mu\int_{\mathscr{P}}\left\{2\varphi\boldsymbol{v}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{d}) + \boldsymbol{v}\cdot\boldsymbol{\nabla}w + \varphi\lambda\nabla^2\lambda + \frac{1}{12}\,\varphi w\nabla^2 w\right\}\mathrm{d}x\,\mathrm{d}y\, +$$

$$+ \int_{\mathscr{P}}\left\{p_a\left(\boldsymbol{v}\cdot\boldsymbol{\nabla}\eta - \lambda - \frac{1}{2}\,w\right) + P\left(\boldsymbol{v}\cdot\boldsymbol{\nabla}h + \lambda - \frac{1}{2}\,w\right)\right\}\mathrm{d}x\,\mathrm{d}y - \boldsymbol{e}_3\cdot\int_{\partial\mathscr{P}}\Pi\boldsymbol{v}\times\mathrm{d}\boldsymbol{r}\,.$$

Now we derive the balance of mass and linear momentum, within the scheme of the columnlike motion, through the invariance of the energy balance equation (2.5) under superposed rigid-body motions. The routine procedure provides

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\varrho\varphi\,\mathrm{d}x\,\mathrm{d}y = 0\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\varrho\varphi\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathscr{P}}[p_a\boldsymbol{\nabla}\eta + P\boldsymbol{\nabla}h + \mu(2\mathrm{d}\boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}w)]\,\mathrm{d}x\,\mathrm{d}y + \boldsymbol{e}_3\cdot\int_{\partial\mathscr{P}}\Pi\,\mathrm{d}\boldsymbol{r}\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\varrho\varphi(\lambda + gt)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathscr{P}}(P - p_a + \mu\varphi\nabla^2\lambda)\,\mathrm{d}x\,\mathrm{d}y\,.$$

On appealing to Green's theorem and to the arbitrariness of $\mathscr{P}$, these equations lead to the corresponding local field equations

$$\dot{\varphi} + \varphi\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0\,, \tag{2.6}$$

$$\varrho\varphi\dot{\boldsymbol{v}} = -\,\boldsymbol{\nabla}\Pi + p_a\boldsymbol{\nabla}\eta + P\boldsymbol{\nabla}h + 2\mu\varphi\boldsymbol{\nabla}\cdot\boldsymbol{d} + \mu\boldsymbol{\nabla}w\,, \tag{2.7}$$

$$\varrho\varphi\dot{\lambda} = P - p_a - \varrho g\varphi + \mu\varphi\nabla^2\lambda\,. \tag{2.8}$$

Therefore, the differential counterpart of eq. (2.5) can be rearranged as the equation

$$\frac{1}{12}\varrho\varphi^2\dot{w} = \Pi - \frac{1}{2}\varphi(P + p_a) + \frac{1}{12}\mu\varphi^2\nabla^2 w\,. \tag{2.9}$$

In spite of the cumbersome structure of the system (2.6)-(2.9), it is worth investigating the column model for two reasons at least. First, it accounts in a natural way for the boundary conditions at the free surface and at the bottom without any particular assumption about the depth function h. Second, although it embodies standard assumptions on the velocity field, which are typical of the shallow-water approximation [17], the column model does not involve any *a priori* condition on the pressure field. Accordingly, such an approach may be viewed as the most general model within the context of shallow-water theories. This assertion, which will be made apparent in next sections, is subtantiated by the feature that, in the case of inviscid fluids, the column model leads straightway to the Korteweg-de Vries equation, while the standard shallow-water theory does not [11,12].

3. – Shallow-water model.

In the shallow-water theory (long-wave approximation) the vertical acceleration of the fluid particles is assumed to be negligible, which is equivalent to identifying the pressure p with the hydrostatic pressure. If viscosity effects are present, such an equivalence no longer holds and, therefore, we have to select the property characterizing the shallow-water theory in viscous fluids.

Following LAMB [18], we start by assuming that the pressure p is, in fact, the hydrostatic pressure, namely

$$p = \varrho g(\eta - z) + p_a\,. \tag{3.1}$$

This in turn implies that

$$P = \varrho g\varphi + p_a\,, \qquad \Pi = (\tfrac{1}{2}\varrho g\varphi + p_a)\varphi\,,$$

thus reducing by two the number of unknown functions. Accordingly, the significant equations are

$$\dot{\varphi} = \varphi\nabla\cdot\boldsymbol{v} = 0\,, \qquad \varrho\varphi\dot{\boldsymbol{v}} = -\,\varrho g\varphi\nabla(\varphi - h) + 2\mu\varphi\nabla\cdot\boldsymbol{d} + \mu\nabla\dot{\varphi}\,,$$

(17) K. O. FRIEDRICHS: *Commun. Pure Appl. Math.*, **1**, 81 (1948).
(18) H. LAMB: *Hydrodynamics*, VI edition (Cambridge, 1932).

which are consistent with the two-dimensional approach carried out before in the literature (see, *e.g.*, [5,7]).

It is worth remarking that, as might be expected, in the viscous shallow-water model the evolution of the horizontal velocity $\boldsymbol{v}$ is affected also by the vertical velocity field $\dot{\varphi}$. Meanwhile, assumption (3.1) does not imply the conditions $\dot{\lambda} = 0$, $\dot{w} = 0$. Indeed, we have

$$\varrho\dot{\lambda} = \mu\nabla^2\lambda, \tag{3.2}$$

$$\varrho\dot{w} = \mu\nabla^2 w\,, \tag{3.3}$$

which may be viewed as compatibility conditions on the solution $\varphi(x, y, t)$, $\boldsymbol{v}(x, y, t)$. Of course, they hold insofar as (3.1) holds. Relations (3.2), (3.3), whereby the velocities λ, w satisfy the diffusion equation, are perfectly consistent with the fact that, usually, accounting for viscosity through Navier-Stokes' law leads to parabolic equations.

4. – Model equations for nonlinear long waves.

Within the scheme pertaining to the column model, the system of equations (2.6)-(2.9) accounts exactly for viscosity and nonlinear inertia terms. Additional restrictions on the scheme allow us to give easily new insights into the properties of water waves in viscous fluids by appealing to a proper stretching transformation. To make this point precise, observe first that the flatness of the bottom, namely $h = h_0$, enables the system (2.6)-(2.9) to be written as

$$\dot{\varphi} + \varphi\nabla\cdot\boldsymbol{v} = 0\,, \qquad \varrho\varphi\dot{\boldsymbol{v}} = -\nabla(\Pi - \varphi p_a) + 2\mu\varphi\nabla\cdot\boldsymbol{d} + \mu\nabla\varphi\,,$$

$$P - p_a = \tfrac{1}{2}\varrho\varphi\ddot{\varphi} + \varrho g\varphi - \tfrac{1}{2}\mu\varphi\nabla^2\dot{\varphi}\,, \qquad \Pi - \varphi p_a = \tfrac{1}{3}\varrho\varphi^2\ddot{\varphi} + \tfrac{1}{2}\varrho g\varphi^2 - \tfrac{1}{3}\mu\varphi^2\nabla^2\dot{\varphi}\,.$$

Now, as usual, suppose that the bottom is flat and that the fields under consideration depend on t and on the spatial co-ordinate x only. Then, denoting by u the x-component of $\boldsymbol{v}$, the relevant equations are

$$\varphi_t + (\varphi u)_x = 0\,, \qquad (\varphi u)_t + (\varphi u^2 + \chi)_x = 2\nu\varphi u_{xx}\,, \tag{4.1}$$

where $\nu = \mu/\varrho$ is the kinematic viscosity, while

$$\chi = \tfrac{1}{3}\varphi^2\ddot{\varphi} + \tfrac{1}{2}g\varphi^2 - \tfrac{1}{3}\nu\varphi^2(\dot{\varphi})_{xx} - \nu\dot{\varphi}\,. \tag{4.2}$$

The fact that system (4.1) cannot be viewed as a particular case of the one investigated by SU and GARDNER [11] leads us to study in detail the behaviour of (4.1) under stretching transformations.

In the linear approximation with viscosity neglected ($\nu = 0$) the system (4.1) yields the equation

$$\varphi_{tt} - gh_0\varphi_{xx} = 0$$

accounting for waves moving to both the left and the right with speed $c_0 = (gh_0)^{\frac{1}{2}}$. Basing on this observation, consider a wave moving to the right with speed c_0 as fundamental solution of (4.1) which makes $x - c_0 t$ the dominant variable. Accordingly, introduce the new space-time co-ordinates ξ, τ defined as

$$\xi = \varepsilon^{\alpha}(x - c_0 t)\,, \qquad \tau = \varepsilon^{\alpha+1} t\,, \tag{4.3}$$

the parameter α being indeterminate as yet. In stretching transformations the parameter ε plays also the role of ordering parameter in formal expansions of the fields around the equilibrium state. Hence we write

$$\varphi = h_0 + \varepsilon\varphi' + \varepsilon^2\varphi'' + \dots\,, \tag{4.4}$$

$$u = 0 + \varepsilon u' + \varepsilon^2 u'' + \dots\,. \tag{4.5}$$

Relative to standard theories, the physical parameter ν represents an additional feature of the fluid. For the sake of generality, we let ν be given the form

$$\nu = \varepsilon^{\beta}\nu_0\,, \tag{4.6}$$

ν_0 being a measure of the strength of the viscosity effects ([19]). Of course, on account of (4.6), the exponent β is expected to be nonnegative; its precise value will be determined later. Now by virtue of (4.2) we have

$$\chi = \chi_0 + \varepsilon\chi' + \varepsilon^2\chi'' + \dots\,, \tag{4.7}$$

where

$$\chi_0 = \tfrac{1}{2} c_0^2 h_0\,, \qquad \chi' = c_0^2\varphi'\,,$$

$$\chi'' = c_0^2\varphi'' + \tfrac{1}{2} g(\varphi')^2 + \tfrac{1}{3} c_0^2 h_0^2 \varepsilon^{2\alpha-1}\varphi'_{\xi\xi} + \nu_0 c_0 \varepsilon^{\alpha+\beta-1}\varphi'_{\xi}\,.$$

We are now in a position to derive the consequences of eqs. (4.1); in terms of ξ and τ, we get

$$\begin{cases} \varepsilon\varphi_{\tau} + (u - c_0)\varphi_{\xi} + \varphi u_{\xi} = 0\,, \\ \varepsilon u_{\tau} + (u - c_0)u_{\xi} + \dfrac{1}{\varphi}\chi_{\xi} = 2\nu_0\varepsilon^{\alpha+\beta}u_{\xi\xi}\,. \end{cases} \tag{4.8}$$

([19]) To admit a dependence of physical parameters on the ordering parameter is customary in the literature (see, *e.g.*, ([3])); an analogous procedure is adopted in connection with particular fields (see, *e.g.*, H. WASHIMI and T. TANIUTI: *Phys. Rev. Lett.*, **17**, 996 (1966)).

Substitute (4.4)-(4.7) into system (4.8); to leading order, both eq. (4.8) are

$$u'_\xi = \frac{c_0}{h_0}\varphi'_\xi ,$$

whence

$$u' = \frac{c_0}{h_0}\varphi' + f(\tau) .$$

The usual asymptotic conditions, namely $u' \to 0$ and $\varphi' \to 0$ for $\xi \to \pm\infty$, allow us to set $f(\tau) = 0$ ([13]).

At next order, eqs. (4.8) take the form

$$\varphi'_\tau + \frac{2c_0}{h_0}\varphi'\varphi'_\xi - (c_0\varphi''_\xi - h_0 u''_\xi) = 0 ,$$

$$\frac{c_0}{h_0}\varphi'_\tau + \frac{c_0^2}{h_0^2}\varphi'\varphi'_\xi + \frac{1}{3}c_0^2 h_0 \varepsilon^{2\alpha-1}\varphi'_{\xi\xi\xi} - \frac{c_0}{h_0}\nu_0\varepsilon^{\alpha+\beta-1}\varphi'_{\xi\xi} + \frac{c_0}{h_0}(c_0\varphi''_\xi - h_0 u''_\xi) = 0 .$$

A direct comparison yields

$$\varphi'_\tau + \frac{3c_0}{2h_0}\varphi'\varphi'_\xi - \frac{\nu_0}{2}\varepsilon^{\alpha+\beta-1}\varphi'_{\xi\xi} + \frac{1}{6}c_0 h_0 \varepsilon^{2\alpha-1}\varphi'_{\xi\xi\xi} = 0 . \tag{4.9}$$

Consistently with our procedure the exponent $\alpha + \beta - 1$ and $2\alpha - 1$ must be nonnegative and hence $\alpha \geqslant 1 - \beta$, $\alpha \geqslant \frac{1}{2}$. Thus three cases are possible. First, $0 \leqslant \beta < \frac{1}{2}$, $\alpha = 1 - \beta$; eq. (4.9) reduces to the Burgers equation ([20])

$$\varphi'_\tau + \frac{3c_0}{2h_0}\varphi'\varphi'_\xi - \frac{\nu_0}{2}\varphi'_{\xi\xi} = 0 . \tag{4.10}$$

Second, $\alpha = \beta = \frac{1}{2}$; this provides the Korteweg-de Vries-Burgers equation ([15])

$$\varphi'_\tau + \frac{3c_0}{2h_0}\varphi'\varphi'_\xi - \frac{\nu_0}{2}\varphi'_{\xi\xi} + \frac{1}{6}c_0 h_0^2\varphi'_{\xi\xi\xi} = 0 . \tag{4.11}$$

Third, $\beta > \frac{1}{2}$, $\alpha = \frac{1}{2}$; eq. (4.9) simplifies to the Korteweg-de Vries equation ([21])

$$\varphi'_\tau + \frac{3c_0}{2h_0}\varphi'\varphi'_\xi + \frac{1}{6}c_0 h_0^2\varphi'_{\xi\xi\xi} = 0 . \tag{4.12}$$

It is a trivial task to obtain the counterparts of (4.10)-(4.12) in terms of the original co-ordinates x, t. For example, letting $n = \varepsilon\varphi'$, eq. (4.11) becomes'

$$n_t + \frac{3c_0}{2h_0}nn_x + c_0 n_x - \frac{\nu}{2}n_{xx} + \frac{1}{6}c_0 h_0^2 n_{xxx} = 0 . \tag{4.13}$$

([20]) J. M. BURGERS: *Adv. Appl. Mech.*, **1**, 171 (1948).
([21]) D. J. KORTEWEG and G. DE VRIES: *Philos. Mag.*, **39**, 422 (1895).

The outstanding result of this section may be phrased by saying that (4.10) or (4.12) accounts for gravity waves in viscous fluids when dissipation (viscosity) or dispersion (inertia) dominates, respectively. If, instead, these phenomena affect gravity waves with comparable strengths, then eq. (4.11) constitutes the appropriate description.

5. – Final remarks.

Starting from the column model approximation, we have derived the balance equations for an incompressible viscous fluid. Then, confining the attention to flat bottoms, we have shown how eq. (4.13) governs the evolution of long water waves. It is worthy of note that eq. (4.13) agrees with the analogous one derived by MEI ([7]). Sometimes, however, it is claimed that viscosity effects are described by other equations ([3,4,22]). Accordingly, we hope that the present paper results in a significant step towards a unified description of viscosity.

In the last decade, considerable attention has been focused on the effects of uneven bottoms on the evolution of long water waves—see, *e.g.*, ([23,24]). It is our intention to hinge again upon the balance equations stated in sect. **2** so as to investigate, through a nonlinear theory, the evolution of solitary waves under the influence of variable depth and viscosity. In a sense, this would represent the counterpart of Djordjevic's paper ([24]) in which viscosity is described via the diffusion operator ([3,4]).

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([22]) W. FERGUSON, P. SAFFMAN and H. YUEN: *Stud. Appl. Math.*, **58**, 165 (1978).
([23]) T. KAKUTANI: *J. Phys. Soc. Jpn.*, **30**, 272 (1971).
([24]) V. D. DJORDJEVIC: *Int. J. Non-Linear Mech.*, **15**, 443 (1980).

● RIASSUNTO

Si considera il comportamento di un liquido viscoso il cui flusso conserva la struttura delle colonne materiali. Si formula un bilancio dell'energia che contiene il termine dissipativo di Navier-Stokes; applicando quindi l'invarianza di tale legge di bilancio per moti rigidi si deducono le leggi di bilancio per la massa e la quantità di moto. È una rilevante conseguenza della teoria che, per la simultanea presenza di termini inerziali e viscosità, le onde lunghe di gravità sono governate dall'equazione combinata di Korteweg-de Vries e Burgers.

Влияние вязкости на поведение гравитационных водяных волн.

Резюме (*). — Рассматривается поведение вязкой жидкости, поток которой сохраняет структуру столба вещества. Формулируется баланс энергии, который содержит диссипативный член Навье-Стокса. Используя инвариантность этого закона относительно недеформируемых движений, выводятся уравнения баланса для массы и импульса. Получено важное следствие этой теории: одновременное появление членов, связанных с вязкостью и инерцией, приводит к тому, что длинные гравитационные волны определяются объединенным уравнением Кортевега-де Вриса и Бургерса.

(*) *Переведено редакцией.*