TWO CONTINUUM APPROACHES TO A WAVELENGTH-DEPENDENT DESCRIPTION OF HEAT CONDUCTION

F. BAMPI and A. MORRO

Istituto di Matematica dell'Università, Genova, Italy

and

D. JOU

Departament de Termologia, Universitat Autònoma de Barcelona, Bellaterra, Barcelona, Spain

Received 14 October 1980

Heat conduction in rigid solids is examined with the purpose of setting up a systematic thermodynamic description accounting for frequency- and wavelength-dependent phenomena. Such a scheme is exhibited in two versions, namely extended irreversible thermodynamics and hidden variable thermodynamics. Though starting from different assumptions, the two versions lead to the same practical conclusions, which in turn may be identified with Chester's microscopic results on high-frequency thermometry. Also a generalization compatible with discontinuity propagation is proposed.

1. Introduction

The classical theory of non-equilibrium thermodynamics[1-3] has been very fruitful in accounting for a wide variety of irreversible processes. However, this theory, based on the local equilibrium hypothesis, is restricted to first-order constitutive equations. In order to get a more general formalism, including both linear and non-linear terms, many theories have been proposed. Amongst those which have received a more recent development we mention the so-called extended irreversible thermodynamics[4-6] and the hidden variable thermodynamics[7-9].

When dealing with non-equilibrium thermodynamics, heat conduction is obviously a basic phenomenon, which motivates the very wide literature on the subject. In spite of this, new physical motivations suggest to look again at heat conduction in the context of non-equilibrium thermodynamics. Indeed, as is well known, the usual continuum approach should be valid in the limit of long wavelengths. However, in presence of discontinuity waves like, for example, shocks or heat pulses, short wavelength oscillations are also excited. Therefore, in order to describe the behaviour of a solid through a continuum theory, we have to improve the standard model possibly by adding new terms, containing time and space derivatives, in the constitutive equations. In this conjunction, owing to the status of the technology of fast-response ther-

mometers and to the increasing interest in the propagation of heat pulses for the spectroscopic analysis of phonon transport in solids[10,11]), investigation of continuum approaches related to relaxation processes and to second sound in solids are no longer purely theoretical in character.

The problem of achieving a more realistic equation accounting for heat conduction in rigid solids has already been considered from a microscopic point of view by Chester[12]) by having recourse to the phonon gas. So, qualitatively Chester obtains an equation for the temperature field which generalizes the one corresponding to the Maxwell–Cattaneo equation in the sense that higher-order space derivatives are involved.

With these observations in mind, our purpose here is twofold. First, to obtain generalizations of the Maxwell–Cattaneo equation where higher-order gradients of the heat flux occur. Indeed, our attempt succeeds both via the extended irreversible thermodynamics (EIT for short) and via the hidden variable thermodynamics (HVT for short). This, in turn, allows us to gain new insights into these two approaches to non-equilibrium thermodynamics. The second purpose is to investigate the connection of our results with those obtained by Chester. Upon establishing such a connection we ascertain a complete agreement and hence we can express our phenomenological coefficients in terms of properties of the phonon energy spectrum.

The plan of the paper is as follows. Heat conduction is investigated in the framework of EIT (section 2) and through the formalism of HVT (section 3). Then, in section 4, the two formalisms are compared while, in section 5, the connection is made with high-frequency thermometry and the microscopic expressions from solid-state theory. Finally, in section 6, a more general approach compatible with wave propagation is outlined.

2. Extended irreversible thermodynamics

Essentially, EIT is a mesoscopic description of many-particle systems whose independent variables are not only the classical variables (the specific internal energy u, in the case of a rigid heat conductor), but also the dissipative fluxes (here the heat flux q). The evolution equation for u is the customary energy balance equation

$$\rho \dot{u} = - \nabla \cdot q + \rho r, \tag{2.1}$$

r being the heat supply per unit mass which we drop out here for the sake of simplicity. Meanwhile, the evolution equation for the heat flux is not known from the start. It is the main problem of EIT to obtain the time derivative of q in terms of u, q, and their spatial derivatives, thus providing an equation for the evolution of q[5,6,13]).

As usual in EIT, we assume that the entropy s is a function not only of u but also of q, and that its differential is given by the generalized Gibbs equation[4-6,13,14])

$$ds = T^{-1} du + T^{-1}\rho^{-1}\alpha_1 q \cdot dq, \qquad (2.2)$$

where α_1 is a function of u, indeterminate as yet, and ρ is the mass density. In view of (2.1) and (2.2) we find that

$$\rho\dot{s} = -\nabla \cdot (T^{-1}q) + q \cdot (\nabla T^{-1} + T^{-1}\alpha_1 \dot{q}). \qquad (2.3)$$

It is one of the basic properties of the entropy s that, in all possible processes, its production σ must be non-negative. The expression of σ may be obtained from its definition in terms of $\rho\dot{s}$ and of the divergence of the entropy flux J_s, namely

$$\sigma = \rho\dot{s} + \nabla \cdot J_s \geq 0. \qquad (2.4)$$

Substitution of (2.3) yields

$$\sigma = -\nabla \cdot (T^{-1}q - J_s) + q \cdot (\nabla T^{-1} + T^{-1}\alpha_1 \dot{q}) \geq 0. \qquad (2.5)$$

It is worth remarking that we make no hypothesis on the particular expression for the entropy flux J_s but, rather, we aim at obtaining J_s from the theory itself as a constitutive relation. So, we are adopting Müller's viewpoint on the entropy flux[15]). Accordingly, we need now an equation relating $\dot{q}$ to u, q, and their spatial derivatives. Then, on account of (2.5), we set

$$\nabla T^{-1} + T^{-1}\alpha_1 \dot{q} = \mu_0 q + \mu_1 \nabla^2 q + \mu_2 \nabla(\nabla \cdot q), \qquad (2.6)$$

where the μ's are phenomenological coefficients. This assumption generalizes previous hypotheses[5,16]) on the dependence of the "thermodynamic force" $\nabla T^{-1} + T^{-1}\alpha_1 \dot{q}$ on the "thermodynamic flux" q. Consistently with the requirements of the second law of thermodynamics, as expressed by the inequality (2.5), we substitute (2.6) into (2.5); then, supposing μ_1 and μ_2 constant, integration by parts provides

$$\sigma = -\nabla \cdot \{T^{-1}q - 2\mu_1(\nabla q)^s q - (\mu_2 - \mu_1)(\nabla \cdot q)q - J_s\}$$
$$+ \mu_0 q^2 - 2\mu_1(\overset{\circ}{\nabla}q)^s : (\overset{\circ}{\nabla}q)^s - (\mu_2 - \tfrac{1}{3}\mu_1)(\nabla \cdot q)^2 \geq 0, \qquad (2.7)$$

where the colon denotes double contraction, the superscript s the symmetric part, and a superposed spot the traceless part of the corresponding tensors.

The expression (2.7) is positive for all imaginable processes, i.e. for all possible fields q, if and only if

$$J_s = T^{-1}q - 2\mu_1(\nabla q)^s q - (\mu_2 - \mu_1)(\nabla \cdot q)q \qquad (2.8)$$

and

$$\mu_0 \geqslant 0, \qquad \mu_1 \geqslant 0, \qquad -\mu_1 + 3\mu_2 \leqslant 0. \tag{2.9}$$

Through the condition (2.8), the theory itself determines the form of the entropy flux which turns out to be different from the usual term $T^{-1}q$ in isotropic rigid conductors[17]) because here we allow the dependence on ∇q.

To clarify the meaning of the parameters α_1 and μ_0 we look at the Maxwell–Cattaneo equation

$$q = -\lambda \nabla T - \tau \dot{q}, \tag{2.10}$$

where λ is the thermal conductivity and τ is the heat relaxation time. We recover this equation as a special case of (2.6) through the identifications $\mu_0 = (T^2\lambda)^{-1}$, $\alpha_1 = -\tau(T\lambda)^{-1}$.

3. Hidden variable thermodynamics

Basically, a material with hidden variables consists of a set of response functions

$$\phi = \phi(y, a) \tag{3.1}$$

and of a function f governing the evolution of the hidden variables a via a differential equation as

$$\dot{a} = f(y, z, a, \nabla a, \nabla\nabla a, \ldots), \tag{3.2}$$

the symbols y and z standing for suitable sets of real (physical) variables. The rigid heat conductor may be assigned the structure of material with hidden variables by identifying y with the absolute temperature θ and z with the temperature gradient $g = \nabla\theta$. The hidden variables a are assumed here to be represented by a vector Λ, while the response functions are identified with the array (ψ, η, q) of the free energy ψ, the entropy η, and the heat flux q.

To go further, we need some assumptions on the evolution function f. Both to avoid inessential formal difficulties and to get a theory providing the most direct generalization of Fourier's law, we choose the function f so as to make the evolution equations into the single vector equation

$$\dot{\Lambda} = \tau^{-1}(g - \Lambda) + \nu_1 \nabla^2 \Lambda + \nu_2 \nabla(\nabla \cdot \Lambda), \tag{3.3}$$

for every fixed particle, $\tau > 0$ being a relaxation time and ν_1, ν_2 phenomenological coefficients.

The set of response functions ϕ must satisfy the restrictions placed by the second law of thermodynamics. Owing to the presence of second-order

derivatives in the evolution equation (3.3), we allow for the existence of an entropy extra-flux N in addition to the usual term $\theta^{-1}q$. The simplest form of N involving the hidden variable Λ and its first derivatives is

$$N = A(\nabla\Lambda)^s\Lambda + B(\nabla \cdot \Lambda)\Lambda;\qquad (3.4)$$

a connection between the phenomenological coefficients A, B, ν_1 and ν_2 will be given shortly.

We adopt as second law of thermodynamics the identical validity of[4,18]

$$\rho\dot\eta + \nabla \cdot (\theta^{-1}q + N) - \rho r\theta^{-1} \geq 0.\qquad (3.5)$$

Then, on account of (2.1) and the relation $\psi = u - \theta\eta$, it follows that the inequality

$$-\rho(\dot\psi + \eta\dot\theta) - \theta^{-1}q \cdot g + \theta\nabla \cdot N \geq 0 \qquad (3.6)$$

must be true identically. According to our constitutive assumptions, the free energy ψ is given by a function of the form $\psi = \psi(\theta, \Lambda)$; hence, in view of (3.3) and (3.4), the entropy inequality (3.6) may be expressed as

$$\begin{aligned}
&-\rho(\psi_\theta + \eta)\dot\theta - (\theta^{-1}q + \rho\tau^{-1}\psi_\Lambda) \cdot g + \rho\tau^{-1}\psi_\Lambda \cdot \Lambda \\
&+ (\theta A\Lambda - \rho\nu_1\psi_\Lambda) \cdot \nabla^2\Lambda + (\theta B\Lambda - \rho\nu_2\psi_\Lambda) \cdot \nabla(\nabla \cdot \Lambda) \\
&+ \rho\theta A(\nabla\Lambda)^s:(\nabla\Lambda)^s + \rho\theta B(\nabla \cdot \Lambda)^2 \geq 0,
\end{aligned}\qquad (3.7)$$

where the subscripts θ and Λ denote partial derivatives.

Our purpose now is to exploit the inequality (3.7) so as to derive the main restrictions placed by the second law of thermodynamics on our constitutive assumptions. To do this, we observe that, as always[18], the present value of the hidden variable $\Lambda(t)$ is independent of the present value of the real variable $g(t)$. Accordingly, as the quantities $\dot\theta(t)$, $g(t)$ may be chosen arbitrarily and independently of each other, we conclude that the inequality (3.7) holds identically if and only if

$$\eta = -\psi_\theta, \qquad q = -\rho\theta\tau^{-1}\psi_\Lambda,\qquad (3.8)$$

and ψ satisfies the compatibility condition

$$\begin{aligned}
&\rho\tau^{-1}\psi_\Lambda \cdot \Lambda + (\theta A\Lambda - \rho\nu_1\psi_\Lambda) \cdot \nabla^2\Lambda + (\theta B\Lambda - \rho\nu_2\psi_\Lambda) \cdot \nabla(\nabla \cdot \Lambda) \\
&+ \rho\theta A(\nabla\Lambda)^s:(\nabla\Lambda)^s + \rho\theta B(\nabla \cdot \Lambda)^2 \geq 0.
\end{aligned}\qquad (3.9)$$

Whenever the function ψ satisfies the inequality (3.9), the response functions η and q, defined by (3.8), are automatically consistent with the second law of thermodynamics in the form (3.8). On the basis of a merely mathematical standpoint, we cannot prefer any function ψ among all those compatible with (3.9). Physical arguments, instead, allow us to select the particular free energy

function ψ given by

$$\psi = \Psi(\theta) + (2\rho\theta)^{-1}\lambda\tau\Lambda \cdot \Lambda, \tag{3.10}$$

where $\Psi(\theta)$ is the usual equilibrium free energy. First, owing to the independence of the quantities $(\nabla\Lambda)^s$, $\nabla(\nabla \cdot \Lambda)$ on the value Λ, account of (3.10) and the identical validity of the inequality (3.9) imply that

$$\theta^2 A = \nu_1\lambda\tau, \qquad \theta^2 B = \nu_2\lambda\tau,$$
$$A \geq 0, \qquad 2A + 3B \geq 0, \qquad \lambda \geq 0. \tag{3.11}$$

Second, eqs. (3.8) deliver

$$\eta = -\Psi_\theta(\theta) + (2\rho\theta^2)^{-1}\lambda\tau\Lambda \cdot \Lambda,$$
$$q = -\lambda\Lambda. \tag{3.12}$$

Eq. $(3.12)_2$ may be regarded as the most natural generalization of Fourier's law. Indeed, when a time independent uniform field g is considered, eq. (3.3) becomes $\Lambda = g$ and therefore $(3.12)_2$ becomes the standard law of heat conduction. This is the main motivation of the assumption (3.10).

It is worth remarking that there is no contradiction between (2.2) and $(3.12)_1$. As shown in ref. 19, the differences are in fact due to the use of the variables u and q instead of θ and Λ.

We end this section by mentioning that a different approach to materials with hidden (internal) variables is delivered in ref. 20.

4. Comparison between the two methods

We are now in a position to make a comparison between EIT and HVT formalisms, which goes beyond the specific equations and points to a sounder understanding of these approaches. For the sake of concision, however, we will concentrate on some specific outstanding topics.

4.1. Independent variables

In EIT the behaviour of the body is described in terms of physical variables u and q only. Both of them have microscopic expressions and they are well defined also in non-equilibrium situations. On the other hand, HVT involves hidden variables, Λ, besides physical variables, θ and g. However, when looking at hidden variables as solutions of the evolution equation HVT involves in fact only physical variables as EIT does.

In addition, hidden variables may be viewed as averaged global parameters which, in an averaged approximate manner, account for macroscopic

phenomena resulting from microscopic processes. This idea is made clear, for example, in two works of Kluitenberg[21,22]) concerning dielectrics.

4.2. Interpretation of the energy balance equation

In both theories the energy balance equation takes the form

$$\rho \dot{u} = -\nabla \cdot q + \rho r, \tag{2.1}$$

where r is the energy supply per unit mass due, for example, to chemical reactions, nuclear reactions, radiation. In EIT r is always supposed to be given and eq. (2.1) determines the evolution of the internal energy u. In HVT, instead, the exploitation of the second law hinges on the possibility of choosing r so as to make $\dot{u}$ arbitrary.

4.3. Second law of thermodynamics

In EIT the second law is assumed in the form

$$\sigma = \rho \dot{s} + \nabla \cdot J_s \geq 0. \tag{2.4}$$

In this expression $\rho \dot{s}$ is given through the generalized Gibbs equation (2.2) while the selection of the entropy flux J_s is closely related to the constitutive equations and therefore it seems logical, as pointed out by Müller[15]), to determine it in the same way as the remaining constitutive equations. If the Gibbs equation used to obtain $\rho \dot{s}$ were the usual local-equilibrium one, this assumption would indeed be restrictive. However, the temperature T is not exactly the local-equilibrium absolute temperature because it contains also some corrections of the order of q^2, which may be determined once α_1 has been fixed. The restriction placed by the energy balance equation (2.1), relating u and q, is taken into account by introducing this equation into (2.2). Finally, the arbitrariness of q and ∇q leads to the desired results.

In HVT the second law is expressed as the generalized Clausius–Duhem inequality

$$\rho \dot{\eta} + \nabla \cdot (\theta^{-1} q + N) - \rho r \theta^{-1} \geq 0, \tag{3.5}$$

allowing for the possible existence of the entropy extra-flux N. No Gibbs equation is introduced and the expression for $\dot{\eta}$ is derived by appealing to the correspondent constitutive equation and to the assumed evolution equation. Meanwhile the consequences of the energy balance equation (2.1) are inherited by (3.5) through the term ρr. Then, the independence of the hidden variable $\Lambda(t)$ of the arbitrary values $\dot{\theta}(t)$ and $g(t)$ provides the sought results.

5. A physical interpretation: high-frequency thermometry

So far, aside from the inequality (2.9), the parameters μ_1 and μ_2 (or ν_1 and ν_2) are left indeterminate by our thermodynamic theory. This freedom is now so used as to show how the heat equation (2.6) accounts for high-frequency thermometry[12]).

First of all, set

$$\mu_1 = -(T^2\lambda)^{-1}r_0^2, \qquad \mu_2 = -(T^2\lambda)^{-1}r_1^2,$$

r_0 and r_1 being the correlation lengths of the transverse and longitudinal perturbations of the heat flux. Upon substitution, eq. (2.6) becomes

$$q = -\lambda\nabla T - \tau\dot{q} + r_0^2\nabla^2 q + r_1^2\nabla(\nabla \cdot q). \tag{5.1}$$

Now, in view of the energy balance equation (2.1), keeping terms only to first order in the deviations from equilibrium yields

$$q = -\lambda\nabla T - \tau\dot{q} + r_0^2\nabla^2 q - \rho c r_1^2\nabla\dot{T}, \tag{5.2}$$

c being the specific heat. By the same token, taking the divergence of (5.1) we find that

$$\nabla^2 T + r_0^2\nabla^4 T + r_1^2\rho c\lambda^{-1}\nabla^2\dot{T} - \rho c\lambda^{-1}\dot{T} - \tau\rho c\lambda^{-1}\ddot{T}$$
$$+ r_0^2\tau\rho c\lambda^{-1}\nabla^2\ddot{T} - r_0^4\rho c\lambda^{-1}\nabla^4\dot{T} - r_0^2 r_1^2\rho c\lambda^{-1}\nabla^4\dot{T} = 0. \tag{5.3}$$

Looking at r_0^2, r_1^2 and τ as (small) coefficients of terms improving the standard theory, it is reasonable to neglect non-linear quantities in r_0^2, r_1^2, and τ. Accordingly, (5.3) reduces to

$$\nabla^2 T + r_0^2\nabla^4 T + r_1^2\rho c\lambda^{-1}\nabla^2\dot{T} - \rho c\lambda^{-1}\dot{T} - \tau\rho c\lambda^{-1}\ddot{T} = 0, \tag{5.4}$$

which is just the equation deduced by Chester[12]) in his analysis of high-frequency thermometry motivated by the experimental possibilities of fast-response thermometers having submicrosecond response times. It is worth mentioning that Chester's deduction, based on a microscopic point of view, concerns a phonon gas described by a kinetic equation of the relaxation type. As a consequence, the coefficients appearing in (5.4) take on the expressions

$$\lambda(\rho c)^{-1} = \tfrac{1}{3}[\langle R^2\rangle(\langle S^{-2}\rangle_{av}\langle R\tau_u^{-1}\rangle)^{-1}] + \tfrac{1}{3}\langle S^2\rangle_{av}\langle R\tau_n\rangle, \tag{5.5}$$

$$\tau = \langle R^2\rangle\langle R\tau_u^{-1}\rangle^{-1}, \tag{5.6}$$

$$r_1^2 = [\langle S^{-2}\rangle_{av}\langle R\tau_u^{-1}\rangle]^{-1}[\tfrac{1}{3}\langle S^2\rangle_{av}\langle S^{-2}\rangle_{av}\langle R\tau_n\rangle\langle R^2\rangle + \tfrac{2}{3}\langle R^3\tau_n\rangle - \tfrac{2}{3}\langle R\rangle\langle R^2\tau_n\rangle], \tag{5.7}$$

$$r_0^2 = \tfrac{1}{5}\langle S^2\rangle_{av}\lambda(\rho c)^{-1}[\langle S^{-2}\rangle_{av}\langle R\tau_u^{-1}\rangle^{-1}][\langle R\rangle\langle R^3\tau_n^2\rangle - \langle R^3\tau_n\rangle\langle R\tau_n\rangle] \tag{5.8}$$

where $\tau_n(\epsilon)$ and $\tau_u(\epsilon)$ are the energy dependent relaxation times correspond-

ing to the normal and the umklapp processes, respectively, ϵ being the phonon energy, while

$$R = \tau_u(\epsilon)[\tau_u(\epsilon) + \tau_n(\epsilon)]^{-1}, \tag{5.9}$$

and

$$\langle S^m \rangle_{av} = (2S_t^{m-3} + S_l^{m-3})(2S_t^{-3} + S_l^{-3})^{-1} \tag{5.10}$$

S_t and S_l being the transverse and longitudinal velocities. Moreover the bracket $\langle A \rangle$ for any function $A(\epsilon)$ stands for the average defined by

$$\langle A \rangle = \left(\int C(\epsilon) A(\epsilon) \, dk \right) \left(\int C(\epsilon) \, dk \right)^{-1}, \tag{5.11}$$

where k is the wavevector of the phonons and $C(\epsilon)$ is the heat capacity per phonon mode of energy ϵ.

Therefore, the theoretical predictions for the heat capcity $C(\epsilon)$ and the relaxation times $\tau_u(\epsilon)$ and $\tau_n(\epsilon)$ lead to the values for $\lambda(\rho c)^{-1}$, τ, r_1^2, and r_0^2 which may be compared with the experimental results obtained through the analysis of the thermal disturbances by means of (5.4). In particular, it may be seen from (5.6) that the relaxation time τ is related to the dominance of normal or umklapp processes. If normal processes are predominant, τ may have large positive values and the corresponding term becomes predominant in (5.4). If, on the contrary, umklapp processes predominate, τ tends to zero ($R \to 0$) and then (5.4) becomes highly diffusive.

6. An approach compatible with wave propagation

Eq. (5.4), derived by Chester through the microscopic analysis of the phonon behaviour and by ourselves via a phenomenological macroscopic theory, suffers from the paradox of infinite speed of propagation of thermal signals. To remedy this deficiency, we consider the general thermodynamic scheme describing dissipative effects in simple fluids. This subject was analyzed in detail both from the viewpoint of EIT[5] and from the viewpoint of HVT[8,9]. Here we recall only that dissipative effects are accounted for through three dissipative fluxes (or three hidden variables) which represent the heat flux q, the scalar viscous pressure p^v, and the viscous pressure tensor $\mathring{P}^v$. The evolution equations for these fluxes, including cross-effect coupling terms, may be written as[5,9]

$$\dot{q} = -\tau_1^{-1}(q + \lambda \nabla T - LT^2 \zeta^{-1} \nabla p^v - MT^2(2\mu)^{-1} \nabla \cdot \mathring{P}^v), \tag{6.1}$$

$$\dot{p}^v = -\tau_0^{-1}(p^v + \zeta \nabla \cdot v - LT\lambda^{-1} \nabla \cdot q), \tag{6.2}$$

$$(\mathring{\boldsymbol{P}}{}^{v})^{\cdot} = -\tau_2^{-1}[\mathring{\boldsymbol{P}}{}^{v} + 2\mu\mathring{\boldsymbol{V}} - MT\lambda^{-1}(\mathring{\boldsymbol{\nabla}}\boldsymbol{q})^{v}], \tag{6.3}$$

v being the velocity and $\boldsymbol{V} = (\boldsymbol{\nabla}v)^{v}$, while the entropy flux turns out to be

$$\boldsymbol{J}_{s} = T^{-1}\boldsymbol{q} + Lp^{v}\boldsymbol{q} + M\mathring{\boldsymbol{P}}{}^{v}\boldsymbol{q}. \tag{6.4}$$

In this scheme it is possible to deal with heat conduction in rigid bodies by setting $\boldsymbol{\nabla} \cdot v = 0$ and $\mathring{\boldsymbol{V}} = 0$. The resulting theory is hyperbolic[9] and hence it permits wave propagation at finite speed. Moreover, under suitable approximations, Chester's theory is easily recovered. Precisely if, following Chester, we assume that the conditions

$$(p^{v})^{-1}\tau_0\dot{p}^{v} \ll 1 \quad \text{and} \quad (\mathring{\boldsymbol{P}}{}^{v}:\mathring{\boldsymbol{P}}{}^{v})^{-1}\tau_2^{2}(\mathring{\boldsymbol{P}}{}^{v})^{\cdot}:(\mathring{\boldsymbol{P}}{}^{v})^{\cdot} \ll 1$$

hold, eqs. (6.2), (6.3) and the hypothesis of rigid motion deliver

$$p^{v} = LT\lambda^{-1}\boldsymbol{\nabla} \cdot \boldsymbol{q} \tag{6.5}$$

$$\mathring{\boldsymbol{P}}{}^{v} = MT\lambda^{-1}(\mathring{\boldsymbol{\nabla}}\boldsymbol{q})^{v}. \tag{6.6}$$

Now, substitution in (6.1) and linearization provide

$$\boldsymbol{q} = -\lambda\boldsymbol{\nabla}T - \tau_1\dot{\boldsymbol{q}} + [L^2T_0^3(\zeta\lambda)^{-1} + \tfrac{1}{6}M^2T_0^3(2\mu\lambda)^{-1}]\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{q})$$
$$+ \tfrac{1}{2}M^2T_0^3(2\mu\lambda)^{-1}\nabla^2\boldsymbol{q}, \tag{6.7}$$

where T_0 is the reference temperature. This equation stands in complete agreement with Chester's constitutive equation (5.4). In passing we note that the linearization procedure complies with the approximations of Chester about the linearity of the collision term with respect to the number of phonons.

7. Conclusions

The classical Fourier's law of heat conduction, while being extraordinarily useful in describing quasi-stationary phenomena, cannot deal with more sophisticated experiments involving the excitation of short wavelength perturbations. Such experiments are, on the other side, very useful in such fields as, for instance, spectroscopy of phonons in solid state physics. So far, equations appropriate for the description of this kind of experiments have been derived from microscopic theories, while macroscopic thermodynamic formalisms have often passed by these problems.

The main concern of this paper has been precisely to show that, via simple macroscopic theories, it is possible to go further into the problem of heat transport in rather extreme, but experimentally accessible, situations and achieve equations accounting for microscopic properties. It seems satis-

factory that heat transport, even in these special conditions, remains in the scope of macroscopic thermodynamics.

Acknowledgements

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