High velocity frame transformations
II: A new approach to Mansouri and Sexl theory

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Abstract: *General frame transformations between inertial observers are simplified by imposing three kinematical conditions. A theorem is proved that such conditions ultimately correspond to a suitable choice of the coordinates in the two frames. Accordingly, the three kinematical conditions do not imply any genuine restriction. A further, restrictive, condition concerning rotational invariance is imposed, so determining the final form of the frame transformation.*

1. Introduction

In the first paper of this series [1], from now on denoted by I, we introduce a projector operator formalism which is here applied to exhibit a new approach to the theory of high velocity frame transformations. The guideline is constituted by the thorough analysis of Mansouri e Sexl [2] about the problem of testing special relativity against the so-called "ether theories" in which there exists a privileged observer and the principle of relativity no longer holds. At this level of generality, it is important to understand the role of every parameter involved in the frame transformation. Hence a first step is that of determining a frame transformation where all the inessential parameters have been eliminated. To do this, Mansouri and Sexl introduce, as reasonable requirements, three kinematical conditions which ultimately correspond to a suitable choice of the coordinates in the two frames. Afterwards they impose that the absolute space is isotropic; this requirement, although "natural", turns out to be a restrictive condition on the transformation.

In order to cast this approach into the context of our paper [3], we reformulate the three kinematical conditions by involving the projection operator formalism developed in [1]. In so doing we are able to prove the theorem that the three kinematical conditions are accounted for by a unique rotation matrix. Instead of the isotropy condition, we prefer to state a principle of rotational invariance of the transformation matrix around the only privileged direction single out by the relative velocity between frames. Step by step we prove that the Mansouri and Sexl theory and our revisited approach are equivalent.

In conclusion we arrive at a final expression for the frame transformation between inertial observers, which depends on three arbitrary functions. The mathematical property of such a transformation and the problem of clock synchronization will be analyzed in the next papers.

In the sequel we freely use the notations and the results proved in I.

2. The Mansouri-Sexl transformation

In ref. [2], Mansouri and Sexl approach the problem of frame transformations by considering two inertial observers: the absolute observer A and a generic observer F. Here, we denote by small (resp. capital) letters the space-time coordinates of A (resp. F) and let $\mathbf{v}$ be the velocity of F with respect to A, namely the velocity measured by A of the points at rest in F. Analogously we denote by $\mathbf{V}$ the velocity of A with respect to F.

As usual in the literature, Mansouri and Sexl too assume that the frame transformation is linear and write the most general linear transformation between A and F – their eq. (6.1) of ref . [2] – in the form:

$$\begin{aligned} T &= at + \varepsilon X + \varepsilon_2 Y + \varepsilon_3 Z \\ X &= b_1 t + bx + b_2 y + b_3 z \\ Y &= d_1 t + d_2 x + dy + d_3 z \\ Z &= e_1 t + e_2 x + e_3 y + ez \end{aligned} \tag{2.1}$$

where all the coefficients are arbitrary functions of $\mathbf{v}$ which are to be determined either by experiment or by theoretical reasoning. For subsequent applications, it is convenient to recall here the notation used in our paper [3], where we proved that the transformation (2.1) can be written in the form

$$X^i = S^i{}_j x^j + W^i t \tag{2.2a}$$

$$T = H_k x^k + Nt \tag{2.2b}$$

Two consequences of (2.2) are immediately obtained by deriving (2.2a) with respect to the time t to get

$$\frac{dX^i}{dT}\frac{dT}{dt} = S^i{}_j \frac{dx^j}{dt} + W^i$$

which, in view of eq. (2.2b),can be written as

$$N\frac{dX^i}{dT} = S^i{}_j \frac{dx^j}{dt} + W^i \tag{2.3}$$

Observe now that, by definition, $\mathbf{V}$ is the velocity of a point P at rest in A, namely of a point P whose coordinates x^i are constant in time. Then, eq (2.3) gives

$$NV^i = W^i \tag{2.4}$$

Analogously, $\mathbf{v}$ is the velocity of a point P at rest in F, namely of a point P whose coordinates X^i are constant in time. Then, eq (2.3) implies

$$S^i{}_j v^j = -NV^i \tag{2.5}$$

In view of such results, transformation (2.2) can be cast into the form

(2.6a) $X^i = S^i{}_j (x^j - v^j t)$

(2.6b) $T = H_k x^k + Nt$

Before proceeding it is convenient to compare eq. (2.1) with eq. (2.2a); it is straightforward matter to get

(2.7) $$\mathbf{S} = \begin{pmatrix} b & b_2 & b_3 \\ d_2 & d & d_3 \\ e_2 & e_3 & e \end{pmatrix}$$

(2.8) $$\mathbf{W} = (b_1, d_1, e_1)$$

Mansouri and Sexl aim at discussing the assumptions that are usually tacitly made on a transformation like (2.1) by explicitly imposing some "natural" requirements of geometrical and kinematical nature. Here we analyse in detail the conditions imposed by them and propose an alternative approach based on the projection operator formalism [1].

Note that they choose the *x* axis parallel to the velocity $\mathbf{v}$ from the start, although they make this fact explicit only when imposing the third kinematical condition. So, their specific results can be achieved in our general contest by letting $\mathbf{v} = (v,0,0)$ which implies that the projectors introduced in I sect. 2 take the form

(2.9) $$\mathbf{\Pi} = \mathbf{P}(\mathbf{v}/v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{P}(\mathbf{y}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{P}(\mathbf{z}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\mathbf{\Omega} = \mathbf{1} - \mathbf{\Pi} = \mathbf{P}(\mathbf{y}) + \mathbf{P}(\mathbf{z}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We remark that we shall use expressions (2.9) only when a specific comparison with the results of Mansouri and Sexl is carried out; otherwise we maintain all our results independent of this particular choice of the spatial coordinates.

3. First kinematical condition

The first kinematical condition imposed in ref. [2] – their eq. (6.2) – is that the X and x axes slide along each other, i.e.

(Kin 1) $\forall t, x: \quad Y = Z = 0 \quad \rightarrow \quad y = z = 0$

On account of (2.1) they get

(3.1) $d_1 = e_1 = 0$; $d_2 = e_2 = 0$

We remark that the same conclusion (3.1) can be arrived at by changing the direction of the row in (Kin 1). Finally, we note that condition (Kin 1) corresponds to the choice of the X axis in the frame F ; so no generality is lost by imposing condition (Kin 1).

Our approach
Our idea is that of exploiting the link (2.5) between the velocities $\mathbf{v}$ and $\mathbf{V}$. To this end change the coordinates in the frame F according to the formula $\hat{X}^k = R^k{}_i X^i$, where $\mathbf{R}$ is a rotation matrix satisfying the standard orthogonality condition $\mathbf{RR}^{\mathrm{T}} = \mathbf{1}$, the suffix T denoting transposition. Choose $\mathbf{R}$ so that

(3.2) $R^k{}_i V^i = \hat{V}^k = -\alpha^2 v^k$

where α is a suitable scalar quantity. Of course the matrix $\mathbf{R}$ is determined up to a rotation which leaves the components v^i unvaried. A first consequence of this choice is that, formally, we have

$$\mathbf{\Pi} = \mathbf{P(v}/v\mathbf{)} = \mathbf{P(\hat{V}}/\hat{V}\mathbf{)}$$

Also, in view of (3.2), applying the matrix $\mathbf{R}$ to both members of eq. (2.5) implies

(3.3) $R^k{}_i S^i{}_j v^j = -N R^k{}_i V^i = N\alpha^2 v^k$

So, on defining the matrix $\hat{S}^k{}_j = R^k{}_i S^i{}_j$ we conclude with the relation

(3.4) $\hat{S}^k{}_j v^j = N\alpha^2 v^k$

This means that the matrix $\hat{S}$ transforms a vector whose components are proportional to v^i into a vector with the same property. In terms of the operator $\mathbf{\Pi}$, for any vector $\mathbf{a}$ we can write eq. (3.4) in the form

$$\hat{S}^k{}_j \Pi^j{}_p a^p = N\alpha^2 \Pi^k{}_p a^p$$

Applying the operator $\mathbf{\Omega}$, since $\mathbf{\Omega\Pi} = \mathbf{0}$ [1] we easily find

$$\Omega^q{}_k \hat{S}^k{}_j \Pi^j{}_p a^p = N\alpha^2 \Omega^q{}_k \Pi^k{}_p a^p = 0$$

and hence, in view of the arbitrariness of $\mathbf{a}$, we conclude that

$$\Omega^q{}_k \hat{S}^k{}_j \Pi^j{}_p = 0 \quad \Leftrightarrow \quad \mathbf{\Omega\hat{S}\,\Pi} = \mathbf{0}$$

As a first conclusion, we can say that it is always possible to choose $\mathbf{S}$ such that

(3.5) $\quad \mathbf{\Omega S \Pi} = \mathbf{0}$

Condition (3.5) can be interpreted as follows. Since $x^i = x_{\|}^i + \Omega^i{}_j x^j$, in view of transformation (2.6a) we can write

$$X^i = S^i{}_j (x_{\|}^j - v^j t) + S^i{}_j \Omega^j{}_p x^p$$

Applying now the operator $\mathbf{\Omega}$ we obtain

$$\Omega^k{}_i X^i = \Omega^k{}_i S^i{}_j \Pi^j{}_p (x_{\|}^p - v^p t) + \Omega^k{}_i S^i{}_j \Omega^j{}_p x^p$$

which, in view of (3.5), yields

(3.6) $\quad \Omega^k{}_i X^i = \Omega^k{}_i S^i{}_j \Omega^j{}_p x^p$

Note now that equation $\Omega^k{}_i X^i = 0$, resp. $\Omega^j{}_p x^p = 0$, selects a straight line parallel to $\mathbf{V}$ in F, resp. parallel to $\mathbf{v}$ in A. Since, in view of (3.6) $\Omega^k{}_i X^i = 0 \quad \Leftrightarrow \quad \Omega^j{}_p x^p = 0$, the two straight lines coincide: hence $\mathbf{V}$ is parallel and opposite to $\mathbf{v}$.

Comparison
Condition (Kin 1) corresponds to the requirement that a vector parallel to $\mathbf{v}$ transforms into a vector which is still parallel to $\mathbf{v}$, which is exactly the content of eq. (3.4). Since eq. (3.4) is equivalent to eq. (3.5), condition (Kin 1) turns out to be equivalent to eq. (3.5).

To obtain the results (3.1) we use the special projectors (2.9) to put eq. (3.5) into the explicit form

$$\mathbf{\Omega S \Pi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b & b_2 & b_3 \\ d_2 & d & d_3 \\ e_2 & e_3 & e \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ d_2 & 0 & 0 \\ e_2 & 0 & 0 \end{pmatrix} = \mathbf{0}$$

hence

$$\mathbf{\Omega S \Pi} = \mathbf{0} \quad \Leftrightarrow \quad d_2 = 0, \; e_2 = 0$$

We remark that Mansouri and Sexl deduced also that the temporal coordinate t cannot enter the eqs (2.1) for the coordinates Y and Z. Here such a condition follows straightforwardly from eqs. (2.4), (2.5), (2.7) and (2.8). Indeed Mansouri and Sexl omit to deduce eq. (2.4) that the velocity $\mathbf{V}$ is proportional to the vector $\mathbf{W}$. Moreover eq. (2.5) asserts that the velocity $\mathbf{V}$ is also proportional to $\mathbf{Sv}$, a quantity that can be easily calculated via eq. (2.7). Since $\mathbf{v} = (v,0,0)$ we have $\mathbf{Sv} = v(b, d_2, e_2)$. So the conclusion is that $\mathbf{W} = (b_1, d_1, e_1) = v(b, d_2, e_2)$ and hence

$$d_1 = 0 \Leftrightarrow d_2 = 0\ ; \quad e_1 = 0 \Leftrightarrow e_2 = 0$$

thus fully recovering eq. (3.1). The other condition

(3.7) $\quad b_1 = vb$

will be useful in discussing the third kinematical condition.

4. Second kinematical condition

The second kinematical condition imposed in ref. [2] – their eq. (6.4) – is that the (X,Z) and (x,z) planes coincide at all times, i.e. the two frames slide along these planes:

(Kin 2) $\quad \forall t,x,z: \quad Y = 0 \quad \to \quad y = 0$

On account of (2.1) they easily get

(4.1) $\quad d_3 = 0$

As for condition (Kin 1), the same conclusion (4.1) can be arrived at by changing the direction of the row in (Kin 2).We note that condition (Kin 2) corresponds to the choice of the Z axis in the frame F, so no generality is lost by imposing condition (Kin 2).

Our approach
Let condition (Kin 1), namely eq. (3.5), hold true. As seen previously, condition (Kin 1) is compatible with rotations around $\mathbf{v}$. Here we fix such degree of freedom.

Define in A two mutually orthogonal unit vectors $\mathbf{y}$ and $\mathbf{z}$ orthogonal to $\mathbf{v}$. In the frame F define the unit vector $\mathbf{Z}$ as

(4.2) $\quad Z^i = \zeta \Omega^i{}_p S^p{}_j z^j$

where $\zeta = 1/|\mathbf{\Omega S z}|$. By definition $\mathbf{Z}$ is orthogonal to $\mathbf{V}$. Perform now a rotation $\mathbf{R}$ around $\mathbf{V}$ (or $\mathbf{v}$, which is the same in view of (Kin 1)) in such a way that

$$R^i{}_q Z^q = z^i$$

note that such rotation does indeed exist because both $\mathbf{z}$ and $\mathbf{Z}$ are unit vectors orthogonal to $\mathbf{V}$ (or equivalently $\mathbf{v}$). Therefore applying the matrix $\mathbf{R}$ to both members of (4.2) we get

$$z^i = R^i{}_k Z^k = \zeta R^i{}_k \Omega^k{}_p S^p{}_j z^j$$

Now, as easily follows from eq. (3.9) of I, we have $\mathbf{R\Omega} = \mathbf{\Omega R}$; so the last relation can be written as

$$z^i = \zeta \Omega^i{}_k R^k{}_p S^p{}_j z^j$$

Applying to both members the projector operator **P(y)** and accounting for the obvious relation $\mathbf{P(y)\Omega} = \mathbf{P(y)}$ we find

$$0 = P(y)^q{}_i z^i = \zeta P(y)^q{}_i \Omega^i{}_k R^k{}_p S^p{}_j P(z)^j{}_r z^r = \zeta P(y)^q{}_i R^i{}_p S^p{}_j P(z)^j{}_r z^r$$

Then we can say that it is always possible to choose $\mathbf{S}$ such that

(4.3) $\qquad \mathbf{YSZ} = \mathbf{0}$

To understand the meaning of condition (4.3) define formally a unit vector $\mathbf{Y}$ by means of the relation

$$Y^i = y^i$$

By definition $\mathbf{Y}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{z}$ and hence to both $\mathbf{V}$ and $\mathbf{Z}$. Multiply eq. (3.6) by **P(y)** to get

$$P(y)^k{}_i X^i = P(y)^k{}_i S^i{}_j \Omega^j{}_p x^p = P(y)^k{}_i S^i{}_j (P(y)^j{}_p + P(z)^j{}_p) x^p = P(y)^k{}_i S^i{}_j P(y)^j{}_p x^p + P(y)^k{}_i S^i{}_j P(z)^j{}_p x^p$$

In view of (4.3) the last term is null. So we arrive at the relation

(4.4) $\qquad P(y)^k{}_i X^i = P(y)^k{}_i S^i{}_j P(y)^j{}_p x^p$

Note now that equation $P(y)^k{}_i X^i = 0$, resp. $P(y)^j{}_p x^p = 0$, selects the plane generated by $\mathbf{V}$ and $\mathbf{Z}$ in F , resp. by $\mathbf{v}$ and $\mathbf{z}$ in A. Since in view of (4.4) $P(y)^k{}_i X^i = 0 \;\Leftrightarrow\; P(y)^j{}_p x^p = 0$, the two planes coincide or, better, they slide on each other.

Comparison
Condition (Kin 2) corresponds to choosing a particular rotation around $\mathbf{v}$ allowing us to select the unit vector $\mathbf{Z}$, which is exactly the content of eq. (4.3).

To obtain the results (4.1) we use the special projectors (2.9) to put eq. (4.3) into the explicit form

$$\mathbf{YSZ} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} b & b_2 & b_3 \\ d_2 & d & d_3 \\ e_2 & e_3 & e \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d_3 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}$$

hence

$$\mathbf{YSZ} = \mathbf{0} \quad \Leftrightarrow \quad d_3 = 0$$

A remark is in order. The procedure we adopted here can be performed independently of condition (Kin 1) provided that all the projector operators are formally written in terms of the velocity $\mathbf{v}$. Indeed The last implication is independent of the results (3.1).

5. Third kinematical condition and comments

The third kinematical condition imposed in ref. [2] – their eq. (6.6) – is that the origin of F moves along the x axis with velocity v with respect to A:

(Kin 3) $\quad x = vt, \quad y = z = 0 \quad \rightarrow \quad X = Y = Z = 0$

On account of (17) they get $b_1 t + bvt = 0$ that is

(5.1) $\quad b_1 = bv$

which coincides with eq. (3.7).

A first conclusion
Having imposed such three kinematical conditions, Mansouri and Sexl arrive at writing the spatial part of transformation (2.1) – their eq. (6.7) – in the form:

$$(5.2) \qquad \begin{aligned} X &= b(x - vt) + b_2 y + b_3 z \\ Y &= dy \\ Z &= ez + e_3 y \end{aligned}$$

Here we proved that the three kinematical conditions are summed up by the fact that we can always choose the matrix $\mathbf{S}$ satisfying the following relations

(5.3) $\quad \mathbf{\Omega S \Pi} = \mathbf{0}; \qquad \mathbf{YSZ} = \mathbf{0}$

This conclusion is made evident by the following

THEOREM 5.1. *There exists a unique rotation matrix $\mathbf{R}$ that makes the transformation matrix $\mathbf{S}$ given by eq. (2.7) into the form*

(5.4) $$\hat{\mathbf{S}} = \mathbf{RS} = \begin{pmatrix} b & b_2 & b_3 \\ 0 & d & 0 \\ 0 & e_3 & e \end{pmatrix}$$

Proof. To prove the theorem it is convenient to use a vector notation. Precisely on introducing the three vectors

$$\mathbf{s}_1 = (b, d_2, e_2) \quad \mathbf{s}_2 = (b_2, d, e_3) \quad \mathbf{s}_3 = (b_3, d_3, e)$$

we can formally write

$$\mathbf{S} = \begin{pmatrix} | & | & | \\ \mathbf{s}_1 & \mathbf{s}_2 & \mathbf{s}_3 \\ | & | & | \end{pmatrix}$$

Analogously, the rotation matrix $\mathbf{R}$ can be written in terms of three unit vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, mutually orthogonal by means of which we can write

$$\mathbf{R} = \begin{pmatrix} - & \mathbf{r}_1 & - \\ - & \mathbf{r}_2 & - \\ - & \mathbf{r}_3 & - \end{pmatrix}$$

in so doing the orthogonality condition for $\mathbf{R}$ is automatically satisfied. We now impose the relevant components of $\hat{\mathbf{S}} = \mathbf{RS}$ to be zero, which, in the actual notation, means

$$\mathbf{r}_2 \cdot \mathbf{s}_1 = \mathbf{r}_3 \cdot \mathbf{s}_1 = \mathbf{r}_2 \cdot \mathbf{s}_3 = 0$$

A straightforward calculation shows that formulas

$$\mathbf{r}_1 = \frac{\mathbf{s}_1}{|\,\mathbf{s}_1\,|} \qquad \mathbf{r}_2 = \frac{\mathbf{s}_1 \times \mathbf{s}_3}{|\,\mathbf{s}_1 \times \mathbf{s}_3\,|} \qquad \mathbf{r}_3 = \mathbf{r}_1 \times \mathbf{r}_2$$

express uniquely the entries of the rotation matrix $\mathbf{R}$ in terms of the entries of the original matrix S. ■

Due to this theorem, we point out that we can arrive at the form (5.4) without loss of generality by simply changing the coordinates in F through a unique rotation. No other entry of $\mathbf{S}$ can be made vanishing in this way. Therefore to proceed further it is necessary to impose assumptions of different nature which, although physically acceptable, are indeed restrictive. We shall discuss this point in the next section.

6. Isotropy and rotational invariance

Isotropy

To fix some other coefficients, Mansouri and Sexl state the following condition

(S1) There is no preferred direction in A

Without exhibiting any detailed calculation, Mansouri and Sexl claim that condition (S1) implies all these relations

(6.1) $e_3 = 0$; $b_2 = b_3 = 0$; $e = d$

In our notation such relations read

(6.2) $\mathbf{ZSY} = \mathbf{0}$; $\mathbf{\Pi S \Omega} = \mathbf{0}$; $\mathbf{\Omega S \Omega} = d\,\mathbf{\Omega}$

Thus, Mansouri and Sexl conclude that the most general frame transformation accounting for all the above results is – their eq. (6.10):

(6.3)
$$\begin{aligned} X &= b(x - vt) \\ Y &= dy \\ Z &= dz \end{aligned}$$

Rotational invariance
As suggested also by Mansouri and Sexl, condition (S1) implies that the only preferred direction, as far as the system F seen from A is concerned, is $\mathbf{v}$. On the other hand, condition (Kin 1) shows that also the direction of $\mathbf{V}$ represents the same preferred direction. In a sense both observers single out the "same" preferred direction. So it seems to be of interest that we analyze the consequences of the following condition

(R1) the matrix $\mathbf{S}$ is invariant under rotation around the direction of $\mathbf{v}$

Let us make condition (R1) mathematically operative. As is well known by the Euler theorem (see, e.g., ref [4]), the matrix $\mathbf{R}$ describes a rotation around $\mathbf{v}$ if and only if

(6.4) $\mathbf{Rv} = \mathbf{v}$

The idea underlying condition (R1) is that transforming rotated vectors by means of $\mathbf{S}$ gives the rotated transformed vectors, i.e.

$$\mathbf{RX} = \mathbf{SR}(\mathbf{x} - \mathbf{v}t)$$

Since, trivially,

$$\mathbf{RX} = \mathbf{RS}(\mathbf{x} - \mathbf{v}t)$$

comparison yields

$$\mathbf{SR}(\mathbf{x}-\mathbf{v}t)=\mathbf{RS}(\mathbf{x}-\mathbf{v}t)$$

which, in view of the arbitrariness of $\mathbf{x}$, yields

$$\mathbf{SR}=\mathbf{RS}$$

Hence condition (R1) is mathematically equivalent to

$$\mathbf{RSR}^{\mathrm{T}}=\mathbf{S} \tag{6.5}$$

The consequences of condition (6.5) have been drawn in paper I – see formula (3.10). On account of (Kin 1) and (Kin 2) we conclude that

$$\mathbf{S}=b\mathbf{\Pi}+d\mathbf{\Omega} \tag{6.6}$$

Hence, in accordance with eq. (6.3), the spatial part of a generic frame transformation depends on two arbitrary functions.

7. The temporal part of Mansouri-Sexl transformation

The previous results show that the transformation (2.1) has been specialized to the following

$$T=a\,t+\varepsilon_k X^k \tag{7.1a}$$

$$X^i=(b\Pi^i{}_j+d\,\Omega^i{}_j)(x^j-v^j t)=(b\Pi^i{}_j+d\,\Omega^i{}_j)x^j-bv^i t \tag{7.1b}$$

which coincide with their eq. (6.14); in passing we note that our eq. (7.1b) emends a refuse present in their eq. (6.14) where the term $v_j x^j$ is written as $v_j X^j$. In order to compare eq. (7.1a) with ours eq. (2.2b), we must substitute eq. (7.1b) into eq. (7.1a). In so doing we get

$$T=a\,t+\varepsilon_k X^k=at+\varepsilon_k(b\Pi^k{}_j+d\,\Omega^k{}_j)x^j-b\varepsilon_k v^k t=(a-b\varepsilon_k v^k)t+(b\,\varepsilon_k^{\parallel}+d\varepsilon_k^{\perp})x^k$$

Comparison with eq. (2.2b) provides the sought results

$$H_k=b\varepsilon_k^{\parallel}+d\varepsilon_k^{\perp}\quad\Leftrightarrow\quad\varepsilon_k=\frac{1}{b}H_k^{\parallel}+\frac{1}{d}H_k^{\perp} \tag{7.2}$$

$$N=a-b\varepsilon_k v^k=a-H_k v^k\quad\Leftrightarrow\quad a=N+H_k v^k=\frac{1}{n} \tag{7.3}$$

note that in eq. (7.3), by means of which the quantity n is defined, we have used eq. (7.2).

8. Comparison with our approach

The approach we followed in ref. [3] consists in starting from the frame transformation (2.2) and in imposing two experimental results: (a) the constancy of the light speed traveling along closed path (Michelson-Morley experiment) and (b) the account for the transverse Doppler effect. Precisely, we have shown that condition (a) leads us to the transformation

(8.1a) $$T = N\,t + H_k x^k$$

(8.1b) $$X^i = \left(\frac{\gamma^2}{n} \Pi^i{}_j + \frac{\gamma}{n} \Omega^i{}_j \right) (x^j - v^j t)$$

where, as usual, $\gamma = 1/\sqrt{1 - v^2/c^2}$. Note that eqs. (8.1) only account for the Michelson-Morley experiment; comparison with eq. (7.1b) yields

(8.2) $$b = \frac{\gamma^2}{n}\,, \quad d = \frac{\gamma}{n}$$

Accounting now also for the transverse Doppler effect is tantamount to setting $n = \gamma$ and hence relations (8.2) reduce to

(8.3) $$b = \gamma\,, \qquad d = 1$$

In this last case the matrix $S^i{}_j$ is identical to that involved in the Lorentz transformation.

Two comments are in order. First, equations (7.3) and (8.2) show that all the three quantities $a(\mathbf{v})$, $b(\mathbf{v})$, $d(\mathbf{v})$ are known as far as the single function $n(\mathbf{v})$ is known. Second, in ref [3] we have proved that condition $n = \gamma$ has also an interesting mathematical meaning, namely that of rendering the determinant of the frame transformation (8.1) unitary, a condition usually imposed on coordinate transformations.

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