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# On the Covariance of First-Order Differential Systems (*).

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**Summary.** — The transformation law for a first-order differential system under Cartesian-coordinate transformation is deduced and the concept of the relevant symmetry group is made precise. Also, it is proved that both the conservative structure and hyperbolicity of a system are preserved. The case of linear elasticity is examined in details.

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## 1. – Introduction.

In the theory of wave propagation, it is usually assumed that the field is represented by an $n$-dimensional column matrix $U$, whose dependence on time $t$ and on Cartesian space coordinates $x_p$, $p = 1, 2, 3$, is determined by a first-order differential system of the form [1]

$$A^0 \frac{\partial U}{\partial t} + A^p \frac{\partial U}{\partial x_p} = B\,; \tag{1.1}$$

the $n \times n$ matrices $A^0$, $A^p$ and the $n$-dimensional column vector $B$ depend on $U$ and, possibly, on $t$ and $x_p$; the summation convention is assumed throughout. It is one of the most remarkable results of the theory that the propagation speeds and the polarization vectors are evaluated algebraically by the sole knowledge of the matrices $A^0$ and $A^p$—see, *e.g.*, [1,2]. The important case of second-order differential systems is embodied into the theory through the result that every second-order system can be cast into an equivalent first-order system by augmenting suitably the number of variables [3]—for a detailed investigation on the link between such two systems see [4].

Suppose now that a differential system describes the behaviour of a physical phenomenon; accordingly, the equations of the system are necessarily tensorial in character. Thus, when the differential system is written in the form (1.1), the entries

(*) The authors of this paper have agreed to not receive the proofs for correction.

of the column matrix $U$ are either scalars or tensors of any rank. A problem arises: to analyse how the column matrix $U$ and the matrices $A^0$ and $A^p$ transform when passing to a different Cartesian-coordinate frame. Indeed the answer to this problem is not immediate in that the entries of $U$ collect together objects possessing different tensorial behaviour.

Therefore, a preliminary result must be deduced which consists in determining and analysing the transformation law for the field $U$ (sect. 2). Then, in sect. 3, we arrive at the transformation law for the quantities $A^0$, $A^p$ and $B$; the case of conservative systems is also investigated. The results so obtained allow us to give a precise definition of the group of invariance for system (1.1). The bidimensional wave equation is exhibited as an example of isotropic system. Section 4 shows that coordinate transformation does not affect hyperbolicity. Finally, in sect. 5 we prove that the symmetry group of the system governing linear elasticity coincides with the crystallographic group of the elastic tensor.

## 2. – $n$-th **order orthogonal matrices.**

Assume that a physical phenomenon is described by the differential system (1.1). This means that we have built up $U$ by suitably rearranging into a column matrix the components of all tensors, which are the unknowns of the problem. Implicitly, this procedure defines a mapping between the index $A = 1, \ldots, n$, which labels the entries $U_A$ of $U$, and the indices which select the components of all unknown tensors. Of course such a correspondence is fixed arbitrarily once for all.

Suppose now that we transform the space coordinates according to the law

$$\hat{x}_i = Q_{ip}\, x_p\,, \tag{2.1}$$

where $Q$ is a $3 \times 3$ orthogonal matrix, namely

$$Q_{ip}\, Q_{jp} = \delta_{ij}\,; \tag{2.2}$$

for further reference, we denote by Orth the group of $3 \times 3$ orthogonal matrices. Because of transformation (2.1), the components of the unknown tensors in the new frame are obtained by acting linearly on the old components. Therefore, also the entries of $U$ must transform linearly; so it is possible to set

$$\hat{U} = \mathcal{Q}\, U\,, \tag{2.3}$$

for a suitable $n \times n$ matrix $\mathcal{Q}$. Since all tensors transform independently of each other, every subset of entries of $U$—corresponding to the components of a single tensor—transforms without involving the other entries. Consequently, the matrix $\mathcal{Q}$ possesses necessarily the following block structure:

$$\mathcal{Q} = \left(\begin{array}{c|c|c|c} M_a & 0 & \ldots & 0 \\ \hline 0 & N_b & \ldots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \ldots & P_c \end{array}\right), \tag{2.4}$$

where the non-vanishing blocks $M_a$, $N_b$, $\ldots$, $P_c$ are suitable square matrices of order

$a, b, \dots, c$, respectively. It is obvious that when we consider scalars the corresponding block in $\mathscr{Q}$ is simply the number 1, whereas in the case of vectors the block coincides with the matrix $Q$ itself. Let us deduce the expression of the block for second-rank tensors.

The components $T_{ij}$ of a second-order tensor $T$ transforms according to the formula

$$\hat{T}_{ij} = R_{ijpq} T_{pq} ,$$

by adopting the convenient notation

$$R_{ijpq} = Q_{ip} Q_{jq} . \tag{2.5}$$

Now the components of $T$, when embodied into $U$, are essentially relabelled by the index $A$ which runs on a suitable subset of $\{1, \dots, n\}$; for convenience we denote such a correspondence by the mnemonic notation $(ij) \leftrightarrow A \subset \{1, \dots, n\}$. In view of (2.2) and (2.5) we have

$$R_{(ij)(pq)} R_{(kl)(pq)} = Q_{ip} Q_{jq} Q_{kp} Q_{lq} = \delta_{ik} \delta_{jl} = \delta_{(ij)(kl)} , \tag{2.6}$$

where, by definition,

$$\delta_{(ij)(kl)} = \begin{cases} 1 & \text{for } (ij) = (kl) , \\ 0 & \text{otherwise} . \end{cases}$$

Therefore, by introducing the correspondence $(ij)(pq) \leftrightarrow AB$, relation (2.6) takes the significant form

$$R_{AC} R_{BC} = \delta_{AB} , \tag{2.7}$$

$R_{AB}$ being the entries of a $9 \times 9$ matrix $R$. With reference to (2.4), the square matrix $R$ is just the block that corresponds in $\mathscr{Q}$ to a second-rank tensor. Finally, on denoting by the superscript T the transpose of a matrix and by $I_a$ the identity matrix of order $a$, relation (2.7) can be written as

$$R R^{\mathrm{T}} = I_9 . \tag{2.8}$$

This means that the matrix $R$ is orthogonal. Moreover, the group structure of Orth is preserved. Indeed, if $Q^1, Q^2 \in \mathrm{Orth}$, then we have

$$R^1_{ijpq} = Q^1_{ip} Q^1_{jq} , \qquad R^2_{ijpq} = Q^2_{ip} Q^2_{jq} ; \tag{2.9}$$

so, on defining $Q = Q^1 Q^2$, in view of (2.5), (2.9), it is an easy matter to check that

$$R_{(ij)(pq)} = R^1_{(ij)(rs)} R^2_{(rs)(pq)} \Leftrightarrow R = R^1 R^2 . \tag{2.10}$$

These results are easily extended to higher-rank tensors.

In conclusion, we are able to prove the following

*Theorem* 2.1. Suppose that the tensorial variables of a physical problem are collected into the column matrix $U$. Then, for every orthogonal matrix $Q$, there exixts a correspondence $Q \to \mathcal{Q}$ such that the $n \times n$ matrix $\mathcal{Q}$ is orthogonal,

$$\mathcal{Q}\, Q^{\mathrm{T}} = I_n \,, \tag{2.11}$$

and the group structure of Orth is preserved, namely if $Q^1 \to \mathcal{Q}^1$, $Q^2 \to \mathcal{Q}^2$, then $Q^1 Q^2 \to \mathcal{Q}^1 \mathcal{Q}^2$.

*Proof.* Note first that choosing $U$ is mathematically equivalent to fix the index correspondence between the components of the tensors and the index $A$ of the entries of $U$. Moreover, since the result (2.8) applies to any block appearing in (2.4), all the matrices $M_a$, $N_b$, …, $P_c$ are orthogonal. Therefore

$$\mathcal{Q}\mathcal{Q}^{\mathrm{T}} = \left(\begin{array}{c|c|c|c} M_a M_a^{\mathrm{T}} & 0 & \dots & 0 \\ \hline 0 & N_b N_b^{\mathrm{T}} & \dots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \dots & P_c P_c^{\mathrm{T}} \end{array}\right) = \left(\begin{array}{c|c|c|c} I_a & 0 & \dots & 0 \\ \hline 0 & I_b & \dots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \dots & I_c \end{array}\right) = I_n \,,$$

which proves (2.11). The second part of the theorem is a straightforward consequence of formula (2.10). □

It is an obvious consequence of (2.11) that the matrix $\mathcal{Q}$ is unimodular, namely

$$\det \mathcal{Q} = \pm 1 \,. \tag{2.12}$$

## 3. – Transformation law for first-order systems.

Consider a first-order differential system of the form (1.1). When the space coordinates transform according to eq. (2.1), we readily check that

$$\frac{\partial}{\partial x_p} = \frac{\partial \hat{x}_i}{\partial x_p} \frac{\partial}{\partial \hat{x}_i} = Q_{ip} \frac{\partial}{\partial \hat{x}_i} \,;$$

hence, in view of (2.3), system (1.1) takes on the form

$$\hat{A}^0 \frac{\partial \hat{U}}{\partial t} + \hat{A}^i \frac{\partial \hat{U}}{\partial \hat{x}_i} = \hat{B} \,,$$

where

$$\hat{A}^0 = \mathcal{Q} A^0 \mathcal{Q}^{\mathrm{T}} \,, \qquad \hat{A}^i = Q_{ip} \mathcal{Q} A^p \mathcal{Q}^{\mathrm{T}} \,, \qquad \hat{B} = \mathcal{Q} B \,. \tag{3.1}$$

It is interesting to analyse what happens in the particular, but important, case of conservative systems. Recall first that a conservative system is a quasi-linear differential system of the form [1]

$$\frac{\partial f^0}{\partial t} + \frac{\partial f^p}{\partial x_p} = B \,, \tag{3.2}$$

where the $n$-dimensional column matrices $f^0$ and $f^p$ are functions of the field $U$. Of course, system (3.2) can be cast into the form (1.1) by setting

$$A^0 = \frac{\partial f^0}{\partial U}, \qquad A^p = \frac{\partial f^p}{\partial U}. \tag{3.3}$$

The following theorem establishes the transformation law for conservative systems.

*Theorem* 3.1. Consider a conservative system written both in the form (3.2) and (1.1). Under the coordinate transformation (2.1), the matrices $A^0$ and $A^p$ transform in accordance with (3.1) if and only if the column vectors $f^0$ and $f^p$ transform as

$$\hat{f}^0 = \mathcal{Q} f^0, \qquad \hat{f}^i = Q_{ip} \mathcal{Q} f^p.$$

*Proof.* For ease in writing, consider the condition involving $f^0$ only; the proofs of the other formulae are in fact identical. On assuming that $\hat{A}^0 = \mathcal{Q} A^0 \mathcal{Q}^{\mathrm{T}}$, relations (2.3) and (3.3) implies

$$\hat{A}^0 = \mathcal{Q} \frac{\partial f^0}{\partial U} \mathcal{Q}^{\mathrm{T}} = \left( \frac{\partial}{\partial U} (\mathcal{Q} f^0) \right) \mathcal{Q}^{\mathrm{T}} = \frac{\partial}{\partial \hat{U}} (\mathcal{Q} f^0);$$

hence $\hat{f}^0 = \mathcal{Q} f^0$.

On the contrary, if we assume that $\hat{f}^0 = \mathcal{Q} f^0$, we get

$$\hat{A}^0 = \frac{\partial f^0}{\partial \hat{U}} = \left( \frac{\partial}{\partial U} (\mathcal{Q} f^0) \right) \mathcal{Q}^{\mathrm{T}} = \mathcal{Q} A^0 \mathcal{Q}^{\mathrm{T}},$$

thus proving the theorem. □

The significant consequence of *theorem* 3.1 is that a conservative system remains conservative under the coordinate transformation (2.1).

The transformation law (3.1) allows us to make precise the meaning of group of invariance associated with a given first-order differential system. Specifically, system (1.1) is invariant under the group $G \subset \mathrm{Orth}$, if for every $Q \in G$,

$$A^0 = \mathcal{Q} A^0 \mathcal{Q}^{\mathrm{T}}, \qquad A^p = Q_{pq} \mathcal{Q} A^p \mathcal{Q}^{\mathrm{T}}, \qquad B = \mathcal{Q} B; \tag{3.4}$$

in particular, system (1.1) is isotropic if relations (3.4) hold for every $Q \in \mathrm{Orth}$. It should be pointed out that relations (3.4) imply also a restriction on the dependence of $A^0$, $A^p$, and $B$ on the field $U$; for instance $A^0$ must satisfy the condition

$$A^0(\mathcal{Q} U, Q_{ip} x_p, t) = \mathcal{Q} A^0(U, x_p, t) \mathcal{Q}^{\mathrm{T}}.$$

As an example of an isotropic system, consider the bidimensional wave equation

$$\frac{\partial^2 \varphi}{\partial t^2} - c^2 \left( \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right) = 0.$$

On defining the column vector

$$U^{\mathrm{T}} = \left( \frac{\partial\varphi}{\partial t}\, \frac{\partial\varphi}{\partial x}\, \frac{\partial\varphi}{\partial y} \right),$$

we can write the equivalent first-order system in the form (1.1), where $A^0$ is the identity matrix and

$$A^1 = \begin{bmatrix} 0 & -c^2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} 0 & 0 & -c^2 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Since the most general $2 \times 2$ orthogonal matrix $Q$ is of the form

$$Q = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \qquad \theta \in [0, 2\pi),$$

we have

$$\mathcal{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix},$$

in that $\partial\varphi/\partial t$ is a scalar, whereas $((\partial\varphi/\partial x)(\partial\varphi/\partial y))$ is a vector. It is an easy matter to verify that relations (3.4) are satisfied for every choice of $\theta$. In conclusion, the equivalent first-order system is isotropic.

## 4. – Hyperbolicity.

For any fixed direction $\boldsymbol{n}$, $|\boldsymbol{n}| = 1$, a characteristic speed $\lambda$ associated to the first-order differential system (1.1) is a solution to the algebraic equation •

$$\det(A^p n_p - \lambda A^0) = 0\,, \tag{4.1}$$

where $n_p$ stands for the component of $\boldsymbol{n}$. For any real $\lambda$, define the polarization vector $\Pi$ as the $n$-dimensional column vector which is a non-trivial solution to the algebraic homogeneous system

$$(A^p n_p - \lambda A^0)\Pi = 0\,.$$

The differential system (1.1) is called hyperbolic in the $t$-direction if [1]

*a*) $\det A^0 \neq 0$,

*b*) the characteristic speeds $\lambda$ are all real,

*c*) the set of the polarization vectors $\Pi$ is a basis for $\boldsymbol{R}^n$.

It is a straightforward matter to show that hyperbolicity is preserved under transformation (2.1). Specifically, owing to the transformation laws (2.3) and (3.1) and on account of (2.11), we have *a*) $\det \hat{A}^0 = \det A^0$ and hence $\det \hat{A}^0 \neq 0$;

moreover $\hat{A}^i \hat{n}_i - \hat{\lambda}\hat{A}^0 = \mathcal{Q}(A^p n_p - \hat{\lambda} A^0)\mathcal{Q}^{\mathrm{T}}$ so that

*b*) $\hat{\lambda} = \lambda$,

*c*) $\hat{\Pi} = \mathcal{Q}\Pi$, and hence the vectors $\hat{\Pi}$ are a basis for $\boldsymbol{R}^n$.

An important class of hyperbolic systems is that of the symmetric systems in the sense of Friedrichs and Lax [5]. Precisely, system (1.1) is symmetric if all the matrices $A^0$ and $A^p$ are symmetric and $A^0$ is positive-definite. Obviously, a symmetric system is hyperbolic; the importance of such a definition is due to the fact that the Cauchy initial-value problem is well posed [5]. Trivially, the transformation (2.1) makes symmetric systems into symmetric systems.

We end this section by pointing out a result concerning isotropic systems. In accordance with (4.1), the characteristic speeds $\lambda$ depend on the propagation direction $\boldsymbol{n}$, besides on $U$, $x_p$, and $t$. However, when system (1.1) is isotropic, the characteristic speeds $\lambda$ are necessarily independent of $\boldsymbol{n}$—cf. [6]. To prove this result, whatever the direction $\boldsymbol{n}$ is, it is always possible to change the coordinates via the transformation (2.1) in such a way that $\hat{x}_1$, say, coincides with the direction $\boldsymbol{n}$. In so doing, formula (4.1) becomes

$$\det(\hat{A}^1 - \hat{\lambda}\hat{A}^0) = 0 .$$

Now we know that $\hat{\lambda} = \lambda$ and that, in view of the isotropy, $\hat{A}^1 = A^1$ and $\hat{A}^0 = A^0$; therefore, for every $\boldsymbol{n}$, the problem of finding the characteristic speeds consists always in solving the same equation

$$\det(A^1 - \lambda A^0) = 0 ,$$

which loses any track of $\boldsymbol{n}$. The conclusion is that, for isotropic systems, the speeds $\lambda$ are independent of $\boldsymbol{n}$.

## 5. – An example: linear elasticity.

It is the purpose of this section to discuss in some detail the property of covariance exhibited by the system governing linear elasticity; in particular, we aim to prove that the invariance group of such a system is the crystallographic group which the elastic tensor belongs to.

To this purpose, we remind the reader that the second-order differential system governing the behaviour of a linear elastic solid is the following [7]:

$$\rho \frac{\partial^2 u_p}{\partial t^2} - \frac{\partial \tau_{pq}}{\partial x_q} = 0 , \tag{5.1}$$

where $\rho$ is the (constant) reference mass density, $u_p$ is the displacement vector, and $\tau_{pq}$ is the stress tensor; as usual, linear elasticity is characterized by the constitutive relation (Hooke law)

$$\tau_{pq} = C_{pqrs} \frac{\partial u_r}{\partial x_s}, \tag{5.2}$$

where the tensor $C_{pqrs}$ denotes the elastic constants.

Now we need to reduce system (5.1) to a first-order system. Thus we first define the variables

$$v_p = \frac{\partial u_p}{\partial t}, \qquad w_{pq} = \frac{\partial u_p}{\partial x_q}$$

and then introduce the 12-dimensional column vector

$$U^{\mathrm{T}} = (v_p \; w_\alpha),$$

where the Greek index runs from 4 to 12 and establishes a relationship $\alpha \leftrightarrow (ab)$ according to the scheme

$$\begin{aligned} &4 \leftrightarrow (11), && 5 \leftrightarrow (12), && 6 \leftrightarrow (13),\\ &7 \leftrightarrow (21), && 8 \leftrightarrow (22), && 9 \leftrightarrow (23),\\ &10 \leftrightarrow (31), && 11 \leftrightarrow (32), && 12 \leftrightarrow (33). \end{aligned}$$

In so doing, system (5.2) takes on the conservative form (3.2) by letting $f^0 = U$ and

$$f^p = -(\Phi_{rp} \; \Psi_{\alpha p}),$$

where

$$\Phi_{rp} = \frac{1}{\rho} \tau_{rp}, \qquad \Psi_{\alpha p} = \Psi_{(ab)p} = v_a \delta_{bp}.$$

In view of relation (3.3), system (5.1) can be cast into the form (1.1), where $A^0$ is the identity matrix, while $A^p$ can be written in a block form as

$$A^p = \left(\begin{array}{c|c} J^p & \Gamma^p \\ \hline D^p & K^p \end{array}\right). \tag{5.3}$$

The entries of the matrices involved in (5.3) are explicitly given by

$$J^p{}_{rs} = \frac{\partial \Phi_{rp}}{\partial v_s} = 0, \qquad K^p{}_{\alpha\beta} = \frac{\partial \Psi_{\alpha p}}{\partial w_\beta} = 0,$$

whereas

$$D^p{}_{\alpha s} = -\frac{\partial \Psi_{\alpha p}}{\partial v_s} \Rightarrow D^p{}_{(ab)s} = -\frac{\partial \Psi_{(ab)p}}{\partial v_s} = -\delta_{as}\delta_{bp};$$

$$\Gamma^p{}_{r\beta} = -\frac{\partial \Phi_{rp}}{\partial w_\beta} \Rightarrow \Gamma^p{}_{r(cd)} = -\frac{\partial \Phi_{rp}}{\partial w_{(cd)}} = -\frac{1}{\rho}\frac{\partial \tau_{rp}}{\partial w_{cd}} = -\frac{1}{\rho} C_{rpcd}.$$

In conclusion, the matrix $A^p$ can be written in the form

$$A^p = \left(\begin{array}{c|c} 0 & \Gamma^p \\ \hline D^p & 0 \end{array}\right).$$

Note that the first three components of $U$ form a vector, while the remaining ones are the components of a second-rank tensor; accordingly, the matrix $\mathcal{Q}$ has the form

$$\mathcal{Q} = \left(\begin{array}{c|c} Q & 0 \\ \hline 0 & R \end{array}\right).$$

In agreement with the transformation laws (3.1), previous formulae show that

$$\hat{D}^i = D^i\,, \qquad \hat{\Gamma}^i_{j(cd)} = -\frac{1}{\rho}\hat{C}_{jicd}\,;$$

therefore, as claimed before, we conclude that the system governing linear elasticity is invariant under the same group which leaves the elastic tensor $C_{ijkl}$ invariant.

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