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What are second order hyperbolic systems?

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1. Introduction

One of the most qualified methods for modeling the physical reality consists in arriving at a suitable differential system which describes, in handy terms, the features of the phenomenon investigated. Especially when transient effects and wave propagation are of importance, hyperbolicity is regarded as an indispensable property of the differential system. As a matter of fact, hyperbolicity is a well established concept for first order differential systems [1, 2, 3]; however, it frequently occurs that the involved equations constitute a second order system for which the literature bears evidence of a lack of an intrinsic definition of hyperbolicity. Also for this reason, sometimes wave propagation is investigated without regard to hyperbolicity [4, 5].

So as to overcome this inconvenience, the overwhelming majority of the investigations concerning hyperbolicity is confined to first order systems; in so doing no generality is lost because of the result that, under suitable conditions, every second order system is equivalent to a first order one—see, e.g., [1, p. 43]. In this optics, the original system plays the marginal role of generating the corresponding first order system, although passing from a second to a first order system can produce new features as, for example, the appearance of zero-speed waves, which are physically irrelevant.

The purpose of this paper is that of investigating the connection between the second order and the equivalent first order system from the point of view of hyperbolicity and wave propagation. On adopting the current view that a second order system is hyperbolic provided that the equivalent first order one is, the main result to emerge from the present work is contained in Theorem 5.7 which gives an intrinsic procedure to check whether a second order system is hyperbolic. In arriving at such a theorem, a number of intermediate results are proved which it is worthwhile to mention. First, it is established that the number of zero-speed waves which appear in passing from a second to a first order system is exactly $d(s-1)$, where d is the number of equations involved and s is the number of space coordinates. Also, the link between the polarization vectors arising from the second order system and the eigenvectors relative to the first order one—see formulas (4.5) and (5.3)—is deduced. Finally, the Appendix contains some algebraic results, not readily found in textbooks.

2. A second order differential system

Consider a physical system $\mathscr{S}$, whose actual state is fully described by the vector field ψ, namely by the knowledge of its components ψ^α, $\alpha = 1, \ldots, d$ as functions of time t and of the space coordinates x^i, $i = 1, \ldots, s$. As it happens in many physical situations, the behavior of $\mathscr{S}$ is supposed to be governed by a second order quasilinear differential system of the form

$$\frac{\partial^2 \psi^\alpha}{\partial t^2} + (A^i)^\alpha_\beta \frac{\partial^2 \psi^\beta}{\partial x_i \, \partial t} + (A^{ij})^\alpha_\beta \frac{\partial^2 \psi^\beta}{\partial x_i \, \partial x_j} = 0, \quad \begin{pmatrix} \alpha, \beta = 1, \ldots, d \\ i, j = 1, \ldots, s \end{pmatrix}, \tag{2.1}$$

where the s matrices A^i and the s^2 matrices A^{ij} can depend on the field ψ.

Let Σ be a moving surface of equation $\varphi(x^1, \ldots, x^s, t) = 0$. As usual we denote by n_i the unit normal to Σ, and by λ the normal speed of propagation [2, 6]. On assuming that the field ψ and its first derivatives are continuous whereas its second derivatives can suffer a jump discontinuity across Σ, the geometric and kinematic compatibility conditions establish the existence of a suitable vector quantity Ψ, the polarization vector, whose components Ψ^α satisfy [6, p. 506]

$$\left[\!\left[\frac{\partial^2 \psi^\alpha}{\partial x_i \, \partial x_j} \right]\!\right] = \Psi^\alpha n_i n_j,$$

$$\left[\!\left[\frac{\partial^2 \psi^\alpha}{\partial x_i \, \partial t} \right]\!\right] = -\lambda \Psi^\alpha n_i,$$

$$\left[\!\left[\frac{\partial^2 \psi^\alpha}{\partial t^2} \right]\!\right] = \lambda^2 \Psi^\alpha,$$

the symbol $[\![\cdot]\!]$ standing for the jump across Σ. On substituting such relations into the system (2.1) we arrive at the algebraic condition

$$((A^{nn})^\alpha_\beta - \lambda (A^n)^\alpha_\beta + \lambda^2 \delta^\alpha_\beta) \Psi^\beta = 0, \tag{2.2}$$

where $A^{nn} = A^{ij} n_i n_j$ and $A_n = A^i n_i$. Equation (2.2) admits non trivial solutions for the variables Ψ^α provided that λ is a characteristic speed, namely a solution to the characteristic condition

$$\det(A^{nn} - \lambda A^n + \lambda^2 I_d) = 0, \tag{2.3}$$

where I_d denotes the identity matrix of order d. A property of the previous algebraic equation (2.3) is stated in the following

Theorem 2.1. If $\det A^{nn} \neq 0$ then Eq. (2.3) does not admit $\lambda = 0$ as a solution, and vice versa. Otherwise,

$$\operatorname{rank} A^{nn} \geq d - m, \tag{2.4}$$

where m is the multiplicity of the root $\lambda = 0$.

Proof. Set, for convenience,

$$M^0 = A^{nn}, \quad M^1 = A^n, \quad M^2 = I_d.$$

Lemma A.4—see Appendix—establishes the validity of the relation

$$\det(A^{nn} - \lambda A^n + \lambda^2 I_d) = \sum_{k_1 \ldots k_d} \lambda^{k_1} \ldots \lambda^{k_d} \det(C_1^{k_1} \ldots C_d^{k_d}),$$

where $k_1, \ldots, k_d$ take values 0, 1, 2 while $C_r^{k_r}$ denotes the r-th column of the matrix M^{k_r}. It is a straightforward matter to realize that the coefficient of λ^0 is exactly the quantity $\det A^{nn}$: therefore $\det A^{nn} \neq 0$ is a necessary and sufficient condition to inhibit the appearance of the root $\lambda = 0$. As to the proof of (2.4), note that, in general, the coefficient of λ^p is a suitable linear combination of minors of A^{nn} whose order is $d - p$ and higher. Consequently, if $\operatorname{rank} A^{nn} = d - r$, then Eq. (2.3) has the form

$$\lambda^r P_{2d-r}(\lambda) = 0,$$

where $P_{2d-r}(\lambda)$ is a polynomial of degree $2d - r$ in λ. The statement of the theorem is now obvious. □

The approach developed so far is good for a rough analysis of wave propagation essentially because we do not know whether the Cauchy problem is well posed or not. A definite answer to this problem is given in the case of first order differential system where it is a prominent result the fact that the Cauchy problem is well posed for strictly hyperbolic and symmetric hyperbolic differential systems [7]. So, our program is that of reducing the second order system to a first order one, imposing the hyperbolicity of the latter system, and analyzing what are the consequences on the original second order system.

3. The equivalent first order system

As is well known, the system (2.1) can be cast into a first order system by suitably augmenting the number of independent variables [1]. Precisely it is possible to prove that the solution to a Cauchy problem for the second order system coincides with the solution to the equivalent Cauchy problem for the first order system; it is evident that there are Cauchy problems for the first order system which do not correspond to any Cauchy problem for the second order system.

In the present case, define the new variables

$$u^\alpha = \frac{\partial \psi^\alpha}{\partial t},$$

$$v_i^\alpha = \frac{\partial \psi^\alpha}{\partial x_i},$$

and collect them into a single column vector U defined by

$$U^T = (u^1 \cdots u^d \quad v_1^1 \cdots v_1^d \cdots v_s^1 \cdots v_s^d), \tag{3.1}$$

where the superscript T denotes transposition. Owing to the presence of the mixed derivative of ψ^α with respect to x_i and t, there is not a unique way of arriving at the expression for the equivalent first order system. Precisely, if we adopt the view that $\partial^2\psi^\alpha/\partial x_i\,\partial t = \partial u^\alpha/\partial x_i$, system (2.1) transforms into the quasilinear first order system

$$\frac{\partial U^A}{\partial t} + (\mathscr{A}^i)^A_B \frac{\partial U^B}{\partial x_i} = 0, \quad A, B = 1, \ldots, D, [D = d(s+1)], \tag{3.2}$$

the $D \times D$ matrix $\mathscr{A}^i$ being given by

$$\mathscr{A}^i = \left(\begin{array}{c|c|c|c} (A^i)^\alpha_\beta & (A^{1i})^\alpha_\beta & \cdots & (A^{si})^\alpha_\beta \\ \hline -\delta^i_1\delta^\alpha_\beta & & & \\ \vdots & & \mathscr{N} & \\ -\delta^i_s\delta^\alpha_\beta & & & \end{array}\right), \tag{3.3}$$

where $\mathscr{N}$ denotes the $sd \times sd$ null matrix.

On the other hand, the choice $\partial^2\psi^\alpha/\partial x_i\,\partial t = \partial v_i^\alpha/\partial t$ corresponds to a different form of the equivalent first order system, namely

$$(\hat{\mathscr{A}}^0)^A_B \frac{\partial U^B}{\partial t} + (\hat{\mathscr{A}}^i)^A_B \frac{\partial U^B}{\partial x_i} = 0, \tag{3.2'}$$

where the $D \times D$ matrices $\hat{\mathscr{A}}^0$, $\hat{\mathscr{A}}^i$ are

$$\hat{\mathscr{A}}^0 = \left(\begin{array}{c|c|c|c} \delta^\alpha_\beta & (A^1)^\alpha_\beta & \cdots & (A^s)^\alpha_\beta \\ \hline 0 & \delta^\alpha_\beta & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & \delta^\alpha_\beta \end{array}\right),$$

$$\hat{\mathscr{A}}^i = \left[\begin{array}{c|c|c|c} 0 & (A^{1i})^\alpha_\beta & \cdots & (A^{si})^\alpha_\beta \\ \hline -\delta^i_1\delta^\alpha_\beta & & & \\ \cline{1-1} \vdots & & \mathscr{N} & \\ \cline{1-1} -\delta^i_s\delta^\alpha_\beta & & & \end{array}\right].$$

The system (3.2′) can be cast into the form (3.2) through a left multiplication by the inverse of the matrix $\hat{\mathscr{A}}^0$. Since the matrix $\hat{\mathscr{A}}^0$ is upper triangular, it is easily recognized that $\det\hat{\mathscr{A}}^0 = 1$ and that its inverse is

$$(\hat{\mathscr{A}}^0)^{-1} = \left[\begin{array}{c|c|c|c} \delta^\alpha_\beta & -(A^1)^\alpha_\beta & \cdots & -(A^s)^\alpha_\beta \\ \hline 0 & \delta^\alpha_\beta & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & \delta^\alpha_\beta \end{array}\right].$$

The equivalence between system (3.2) and (3.2′) follows from the relation

$$(\hat{\mathscr{A}}^0)^{-1}\hat{\mathscr{A}}^i = \mathscr{A}^i.$$

As a conclusion, it is not restrictive to consider, from now on, the system (3.2) as the first order equivalent system to (2.1).

In the first order formulation too, it is possible to look at solutions whose first derivatives may suffer jump discontinuities across the surface Σ. The relevant geometric and kinematic compatibility conditions now read [6]

$$\left[\!\left[\frac{\partial U^A}{\partial x_i}\right]\!\right] = \Pi^A n_i,$$

$$\left[\!\left[\frac{\partial U^A}{\partial t}\right]\!\right] = -\lambda\Pi^A,$$

where Π is a suitable column vector. Accordingly, it is possible to deduce the following algebraic condition

$$((\mathscr{A}^n)^A_B - \lambda\delta^A_B)\Pi^B = 0, \tag{3.4}$$

where $\mathscr{A}^n = \mathscr{A}^i n_i$. Nontrivial solutions to Eq. (3.4) for the variables Π^A are ensured by the characteristic condition

$$\det(\mathscr{A}^n - \lambda I_D) = 0. \tag{3.5}$$

Hyperbolicity is tantamount to the claim that Eq. (3.5) admits D real eigenvalues $\lambda_{(1)}, \ldots, \lambda_{(D)}$, not necessarily distinct, and that the corresponding set of eigenvectors $\Pi_{(1)}, \ldots, \Pi_{(D)}$ is a basis for $\mathbb{R}^D$.

4. Zero-speed waves and polarization vectors

The present task is that of relating the two formulations of the same problem by the point of view of the hyperbolicity. Specifically, we want to derive a precise link between the characteristic speeds and between the polarization vectors Ψ and the eigenvectors Π.

Consider first the characteristic speeds which are the solutions to the algebraic equations (2.3) and (3.5), respectively. Owing to the expression (3.3) for the matrix $\mathscr{A}^i$, we can write

$$\mathscr{A}^n - \lambda I_D = \left(\begin{array}{c|c|c|c} (A^n)^\alpha_\beta - \lambda\delta^\alpha_\beta & (A^{1n})^\alpha_\beta & \cdots & (A^{sn})^\alpha_\beta \\ \hline -n_1\delta^\alpha_\beta & \multicolumn{3}{c}{} \\ \cline{1-1} \vdots & \multicolumn{3}{c}{-\Lambda} \\ \cline{1-1} -n_s\delta^\alpha_\beta & \multicolumn{3}{c}{} \end{array}\right), \tag{4.1}$$

where Λ is a $sd \times sd$ diagonal matrix whose diagonal elements are all equal to the quantity λ. Owing to the structure of (4.1), Theorem A.3 is applicable. Comparison of (4.1) with (A.4) shows that, also in the present case, S corresponds to a matrix of order d; hence formula (A.3) yields

$$\det(\mathscr{A}^n - \lambda I_D) = (-\lambda)^{d(s-1)} \det(A^{nn} - \lambda A^n + \lambda^2 I_d). \tag{4.2}$$

Equation (4.2) shows that passing from the second order system (2.1) to the equivalent first order system (3.2) causes the appearance of zero-speed waves, whose number is $d(s-1)$; it is evident that the material waves associated to such speeds have a purely formal meaning.

Look now at the link between the polarization vectors Ψ and the eigenvectors Π. In accordance with (3.1), for convenience we set

$$\Pi^T = (\hat{u}^1 \cdots \hat{u}^d \quad \hat{v}^1_1 \cdots \hat{v}^d_1 \cdots \hat{v}^1_s \cdots \hat{v}^d_s). \tag{4.3}$$

In view of (4.1) and (4.3), the algebraic system (3.4), which determines the eigenvectors Π, explicitly reads

$$\begin{cases} ((A^n)^\alpha_\beta - \lambda\delta^\alpha_\beta)\hat{u}^\beta + (A^{in})^\alpha_\beta \hat{v}^\beta_i = 0, \\ -n_i\hat{u}^\alpha - \lambda\hat{v}^\alpha_i = 0. \end{cases} \tag{4.4}$$

To solve system (4.4), we assume first that $\lambda \neq 0$. Calculating $\hat{v}^\alpha_i$ from $(4.4)_2$ and substituting into $(4.4)_1$ show that the quantities $\hat{u}^\alpha$ satisfy Eq. (2.2). This enables us to choose

$$\hat{u}^\alpha = \lambda\Psi^\alpha,$$

whereby, in view of (4.3) and $(4.4)_2$,

$$\Pi = \begin{pmatrix} \lambda\Psi^\alpha \\ -n_i\Psi^\alpha \end{pmatrix}. \tag{4.5}$$

Suppose now that $\lambda = 0$. In this case, condition $(4.4)_2$ implies $\hat{u}^\alpha = 0$; as a consequence, the restrictions on the components (4.3) of Π are provided by $(4.4)_1$,

$$(A^{in})^\alpha_\beta \hat{v}^\beta_i = 0. \tag{4.6}$$

5. The concept of hyperbolicity for second order systems

The concept of first order hyperbolic systems is well established and most of their properties are well understood [2, 3]. It is then natural to define second order hyperbolic systems in accordance with the following

Definition 5.1. A second order differential system is hyperbolic when the corresponding first order system is hyperbolic in the usual sense.

The precise link between second order and the corresponding first order system, deduced previously, allows us to perform a step-by-step analysis of the consequence of Definition 5.1. To this end we need to prove some preliminary results. On setting

$$\mathscr{A} = (A^{1n} \cdots A^{sn}), \tag{5.1}$$

we have

Lemma 5.1. If, for a second order system, the multiplicity of the characteristic speed $\lambda = 0$ is m, then hyperbolicity implies that

$$\operatorname{rank} \mathscr{A} = d - m. \tag{5.2}$$

Proof. If the second order system possesses the root $\lambda = 0$ with multiplicity m, then we proved in Sect. 4 that the equivalent first order system possesses the root $\lambda = 0$ with multiplicity $d(s-1) + m$. Hyperbolicity implies that $d(s-1) + m$ eigenvectors Π are associated with $\lambda = 0$ whose expressions are to be evaluated via Eq. (4.6). Since the quantities $\hat{v}^\alpha_i$ are ds in number, the previous requisite is satisfied provided that $\operatorname{rank} \mathscr{A} = ds - [d(s-1) + m]$, as indicated in Eq. (5.2). □

From Theorem A.5 and Lemma 5.1 it follows that

$$\operatorname{rank} A^{nn} \leq \operatorname{rank} \mathscr{A} \Rightarrow \operatorname{rank} A^{nn} \leq d - m.$$

Comparing this result with Theorem 2.1 we gain the following result.

Theorem 5.2. For a second order hyperbolic system the rank of the matrix A^{nn} is exactly $d - m$, where m is the multiplicity of the characteristic speed $\lambda = 0$.

The number of the polarization vectors is fixed by the following

Theorem 5.3. For a second order hyperbolic system any characteristic speed λ with multiplicity m is associated with m linearly independent polarization vectors Ψ.

Proof. In the case $\lambda \neq 0$, Eq. (4.5) readily shows that the linear independence of the Π's is mathematically equivalent to the linear independence of the Ψ's. On the other hand, the polarization vectors associated with the characteristic speed $\lambda = 0$ are determined by Eq. (2.2), which now reads

$$(A^{nn})^{\alpha}_{\beta} \Psi^{\beta} = 0.$$

It is a consequence of Theorem 5.2 that there are exactly m independent polarization vectors Ψ associated with the m roots $\lambda = 0$. $\square$

Since the polarization vectors Ψ belong to a d dimensional vector space, Theorem 5.3 shows the validity of the following

Corollary 5.4. A second order hyperbolic system cannot possess more than d coinciding characteristic speeds.

Theorem 5.3 implies a suggestive form for m eigenvectors Π associated with the m roots $\lambda = 0$. Indeed, it is possible to write m solutions to Eq. (4.6) in the form

$$\hat{v}^{\alpha}_{i} = n_i \Psi^{\alpha},$$

where Ψ^{α} denotes the components of any of the m linearly independent polarization vectors associated with the m roots $\lambda = 0$. Accordingly, we can write

$$\Pi = \begin{pmatrix} 0 \\ -n_i \Psi^{\alpha} \end{pmatrix}. \tag{5.3}$$

In conclusion, the expression (4.5) is valid for all the $2d$ eigenvectors Π which correspond to the $2d$ values of the characteristic speeds relative to the second order system.

Since the eigenvectors Π, associated with a first order hyperbolic system, are linearly independent, we can prove the following

Theorem 5.5. The polarization vectors $\Psi_{(p)}, p = 1, \ldots, 2d$, associated with a second order hyperbolic system, span $\mathbb{R}^d$.

Proof. Assume that the first $2d$ eigenvectors $\Pi_{(p)}$, relative to the associated first order system, take the form (4.5). Hyperbolicity implies that

$$\sum_{p=1}^{2d} \kappa_p \Pi_{(p)} = 0 \Leftrightarrow \kappa_p = 0. \tag{5.4}$$

Suppose now that the polarization vectors $\Psi_{(p)}$ do not span $\mathbb{R}^d$, namely there are only $c < d$ linearly independent $\Psi_{(q)}$, $q = 1, \ldots, c$. Accordingly, there exists a suitable set of coefficients P_{rq} such that

$$\Psi_{(r)} = \sum_{q=1}^{c} P_{rq} \Psi_{(q)}, \quad r = c+1, \ldots, 2d. \tag{5.5}$$

On account of (4.5), the system for the linear independence of $\Pi_{(p)}$ explicitly becomes

$$\begin{cases} \displaystyle\sum_{p=1}^{2d} \kappa_p \lambda_{(p)} \Psi_{(p)} = 0, \\ \displaystyle\sum_{p=1}^{2d} \kappa_p \Psi_{(p)} = 0, \end{cases} \tag{5.6}$$

where $\lambda_{(p)}$ is the characteristic speed associated with the polarization vector $\Psi_{(p)}$. In view of (5.5), condition (5.6) takes the form

$$\begin{cases} \displaystyle\sum_{q=1}^{c} \kappa_q \lambda_{(q)} \Psi_{(q)} + \sum_{r=c+1}^{2d} \kappa_r \lambda_{(r)} \left(\sum_{q=1}^{c} P_{rq} \Psi_{(q)} \right) = 0, \\ \displaystyle\sum_{q=1}^{c} \kappa_q \Psi_{(q)} + \sum_{r=c+1}^{2d} \kappa_r \left(\sum_{q=1}^{c} P_{rq} \Psi_{(q)} \right) = 0. \end{cases}$$

The linear independence of $\Psi_{(q)}$ implies

$$\begin{cases} \kappa_q \lambda_{(q)} + \displaystyle\sum_{r=c+1}^{2d} \kappa_r \lambda_{(r)} P_{rq} = 0, \\ \kappa_q + \displaystyle\sum_{r=c+1}^{2d} \kappa_r P_{rq} = 0, \end{cases} \tag{5.7}$$

where the index $q = 1, \ldots, c$ is not summed. It is evident that (5.7) is a system of $2c$ equations in the $2d > 2c$ unknowns κ_p: the system is evidently underdetermined which violates condition (5.4). □

Indeed, the linear independence of the eigenvectors $\Pi_{(p)}$ implies not only Theorem 5.5 but also a more involved condition. To make this claim explicit, we point out that Theorem 5.5 requires $c = d$. In other words, the set of polarization vectors can be divided into two sets: the elements of the first set are the d linearly independent polarization vectors, which will be denoted by the symbol $\Psi_{(\alpha)}$, and the elements of the second set are the remaining d polarization vectors, which will be denoted by the symbol $\Xi_{(\alpha)}$. With this notation Eq. (5.5) reads

$$\Xi_{(\alpha)} = \sum_{\beta=1}^{d} P_{\alpha\beta} \Psi_{(\beta)},$$

where $P_{\alpha\beta}$ is now a $d \times d$ matrix. Also, we denote by $\lambda_{(\alpha)}$ the characteristic speeds associated with $\Psi_{(\alpha)}$ and by $\gamma_{(\alpha)}$ the characteristic speeds associated with $\Xi_{(\alpha)}$. This notation allows us to re-write system (5.7) in the form

$$\begin{cases} k_\alpha \lambda_{(\alpha)} + \sum_{\beta=1}^{d} h_\beta \lambda_{(\beta)} P_{\beta\alpha} = 0, \\ k_\alpha + \sum_{\beta=1}^{d} h_\beta P_{\beta\alpha} = 0, \end{cases} \tag{5.8}$$

where the unknowns κ_p are rearranged according to the relation

$$(\kappa_1, \ldots, \kappa_d, \kappa_{d+1}, \ldots, \kappa_{2d}) = (k_1, \ldots, k_d, h_1, \ldots, h_d).$$

The system (5.8) can be written conveniently in matrix form as follows. Define the diagonal matrices

$$\Lambda = \begin{bmatrix} \lambda_{(1)} & & \\ & \ddots & \\ & & \lambda_{(d)} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \gamma_{(1)} & & \\ & \ddots & \\ & & \gamma_{(d)} \end{bmatrix},$$

and the row vectors

$$K = (k_1 \cdots k_d), \quad H = (h_1 \cdots h_d).$$

Then, system (5.8) becomes

$$(K \quad H)\begin{pmatrix} \Lambda & I_d \\ \Gamma P & P \end{pmatrix} = 0. \tag{5.9}$$

The linear system (5.9) admits only trivial solutions, as required by condition (5.4), if and only if

$$\det\begin{pmatrix} \Lambda & I_d \\ \Gamma P & P \end{pmatrix} \neq 0.$$

On account of Theorem A.3, we have thus proved the following

Theorem 5.6. For a second order hyperbolic system the relation

$$\det(\Gamma P - P\Lambda) \neq 0 \tag{5.10}$$

holds true.

We are now in a position of stating a complete characterization of hyperbolicity for second order system.

Theorem 5.7. On using the notation of this section, a second order differential system (2.1) is hyperbolic if and only if

(i) the characteristic speeds, solutions to (2.3) are all real;
(ii) any characteristic speed with multiplicity m is associated to m linearly independent polarization vectors, solutions to (2.2);
(iii) the $2d$ polarization vectors span $\mathbb{R}^d$;
(iv) the matrix $\Gamma P - P\Lambda$, involved in (5.10), is non-singular;
(v) if the system (2.1) admits the characteristic speed $\lambda = 0$ with multiplicity m, then the matrix $\mathscr{A}$, defined in (5.1), is such that rank $\mathscr{A} = d - m$.

Proof. That conditions (i)–(v) are necessary has been proved already. As to their sufficiency we observe that conditions (i)–(iii) give a meaning to the matrix $\Gamma P - P\Lambda$. Condition (iv) implies the linear independence of the $2d$ eigenvectors relative to the corresponding first order system and associated to the polarization vectors via (4.5). Finally, condition (v) guarantees the linear independence of the eigenvectors which correspond to the $d(s-1)$ zero-speed waves which unavoidably appear in passing from the second order to the first order system. As a consequence the associated first order system is hyperbolic thereby implying, through Definition 5.1, the hyperbolicity of system (2.1). □

An interesting remark to Theorem 5.7 is provided by the following

Theorem 5.8. Condition (v) of Theorem 5.7 is automatically satisfied when the second order system does not allow for vanishing characteristic speeds.

Proof. Owing to Theorem 2.1, the absence of vanishing characteristic speeds is mathematically equivalent to the condition $\det A^{nn} \neq 0$, namely rank $A^{nn} = d$. It is a consequence of Theorem A.5 that rank $\mathscr{A} = d$ since the rank of $\mathscr{A}$ cannot exceed the number d. Hence condition (v) is trivially satisfied. □

6. Comments and conclusions

Theorems 5.7 gives a complete and definite answer to the question posed by the title of this paper. In particular it allows us to identify a second order hyperbolic system without having explicit recourse to the associated first order system. However, whereas conditions (i)–(iv) involve essentially the characteristic speeds and the polarization vectors of the second order system, condition (v) is more peculiar because it keeps track of the fact that the associated first order system possesses unavoidably $d(s-1)$ zero-speed waves besides the characteristic speeds of the second order system. As a consequence, it is possible to design second order systems which are not hyperbolic simply because condition (v) is violated. An example of such systems is provided by the equations

$$\begin{cases} \dfrac{\partial^2\psi^1}{\partial t^2} - a^2\left(\dfrac{\partial^2\psi^1}{\partial x^2} + \dfrac{\partial^2\psi^1}{\partial y^2}\right) = 0, \\ \dfrac{\partial^2\psi^2}{\partial t^2} + f\dfrac{\partial^2\psi^2}{\partial x\,\partial t} + g\dfrac{\partial^2\psi^2}{\partial y\,\partial t} - a^2\left(r\dfrac{\partial^2\psi^1}{\partial x^2} + s\dfrac{\partial^2\psi^1}{\partial y^2}\right) = 0, \end{cases} \tag{6.1}$$

where a, f, g, r, and s are functions of ψ^1 and ψ^2. Comparison with (2.1) yields

$$A^{11} = \begin{pmatrix} -a^2 & 0 \\ -ra^2 & 0 \end{pmatrix}, \quad A^{22} = \begin{pmatrix} -a^2 & 0 \\ -sa^2 & 0 \end{pmatrix}, \quad A^{12} = 0;$$

$$A^1 = \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix}.$$

As a consequence we have

$$A^{nn} = \begin{pmatrix} -a^2 & 0 \\ -a^2 z & 0 \end{pmatrix}, \quad A^n = \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix},$$

having used the shorthands

$$h = fn_1 + gn_2, \quad z = rn_1^2 + sn_2^2.$$

It is a straightforward matter to ascertain that the characteristic speeds are

$$\lambda_{(1)} = 0, \quad \lambda_{(2)} = h, \quad \lambda_{(3)} = a, \quad \lambda_{(4)} = -a,$$

and that the associated polarization vectors take the form

$$\Psi_{(1)} = \Psi_{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \Psi_{(3)} = \begin{pmatrix} c_+ \\ 1 \end{pmatrix}, \quad \Psi_{(4)} = \begin{pmatrix} c_- \\ 1 \end{pmatrix},$$

where

$$c_\pm = -\frac{a^3 \pm h}{az}.$$

It is matter of calculation to prove that conditions (i)–(iv) are satisfied, whereas condition (v) is violated provided that $r \neq s$. Of course, this means that the first order system associated to (6.1) is not hyperbolic.

We conclude this paper by showing that, in the case of a single second order equation, Theorem 5.7 reduces to the usual definition of hyperbolicity [1]. In detail, consider the equation

$$\frac{\partial^2 \psi}{\partial t^2} + 2A^i \frac{\partial^2 \psi}{\partial x_i\, \partial t} + A^{ij} \frac{\partial^2 \psi}{\partial x_i\, \partial x_j} = 0; \tag{6.2}$$

it is evident that Eq. (6.2) represents the most general form for a second order equation, setting aside lower order terms, which are inessential in this context. Trivially, we have $d = 1$, hence the two polarization vectors necessarily coincide. As a consequence, on denoting by $\lambda_{(1)}$ and $\lambda_{(2)}$ the two characteristic speeds, condition (iv) implies that

$$\lambda_{(1)} \neq \lambda_{(2)}. \tag{6.3}$$

Since the characteristic condition (2.3) reads

$$\lambda^2 - 2A^n \lambda + A^{nn} = 0, \tag{6.4}$$

relation (6.3) and the reality of the two characteristic speeds is mathematically equivalent to

$$(A^n)^2 - A^{nn} > 0,$$

or, explicitly,

$$n_i n_j (A^i A^j - A^{ij}) > 0.$$

Owing to the arbitrariness of the direction n_i, we arrive at the result that the matrix $A^i A^j - A^{ij}$ must be positive definite. All these conditions represent the usual definition of second order hyperbolic equations.

A. Appendix

Recall first the following

Lemma A.1. Consider a $D \times D$ matrix written in the bordered form

$$\left(\begin{array}{c|c} M & C \\ \hline R & \lambda \end{array}\right),$$

where M is a $(D-1) \times (D-1)$ matrix, C is a $(D-1)$ column matrix, R is a $(D-1)$ row matrix, and $\lambda \neq 0$ is a real number. There holds the relation

$$\det\left(\begin{array}{c|c} M & C \\ \hline R & \lambda \end{array}\right) = \lambda \det\left(M - \frac{CR}{\lambda}\right). \tag{A.1}$$

The proof of this result is standard—see, e.g., [8, p. 265]. For convenience, denote by $\operatorname{ord} M$ the order of the square matrix M. Formula (A.1) can conveniently be re-written in accordance with the following

Corollary A.2. There holds the relation

$$\det\left(\begin{array}{c|c} M & C \\ \hline R & \lambda \end{array}\right) = \lambda^{1-\operatorname{ord} M} \det(\lambda M - CR).$$

Suppose now that a $D \times D$ matrix has the special (bordered) form

$$\left(\begin{array}{c|c} S & P \\ \hline Q & \Lambda \end{array}\right), \tag{A.2}$$

where Λ is a square diagonal matrix whose diagonal elements are all equal to the nonvanishing number λ. In this case Corollary A.2 implies the following

Theorem A.3. There holds the relation

$$\det\left(\begin{array}{c|c} S & P \\ \hline Q & \Lambda \end{array}\right) = \lambda^{\operatorname{ord}\Lambda - \operatorname{ord} S} \det(\lambda S - PQ). \tag{A.3}$$

If $\operatorname{ord}\Lambda - \operatorname{ord} S \geq 0$, formula (A.3) holds also when $\lambda = 0$.

Proof. On letting $d = \operatorname{ord} S$, write the matrix (A.2) with the following index notation

$$\left(\begin{array}{c|c} S_{\alpha\beta} & P_{\alpha j} \\ \hline Q_{i\beta} & \lambda\delta_{ij} \end{array}\right), \quad \left(\begin{array}{c} \alpha, \beta = 1, \ldots, d \\ i, j = 1, \ldots, \operatorname{ord}\Lambda \\ d + \operatorname{ord}\Lambda = D \end{array}\right), \tag{A.4}$$

and apply Corollary A.2. Since for the matrix (A.4) we have

$$C = \begin{bmatrix} P_{\alpha s} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad R = (Q_{s\beta} \quad 0 \cdots 0),$$

where the number of zeros is equal to $\operatorname{ord}\Lambda - 1$, Corollary A.2 implies

$$\det\left(\begin{array}{c|c} S & P \\ \hline Q & \Lambda \end{array}\right) = \lambda^{1-(D-1)} \det\left(\begin{array}{c|c} S_{\alpha\beta} - P_{\alpha s}Q_{s\beta} & P_{\alpha j} \\ \hline Q_{i\beta} & \lambda\delta_{ij} \end{array}\right),$$

where now is understood that the indices i and j run from 1 to $d-1$. Applying this procedure as many times as the order of Λ provides the required result. The final part of the theorem follows trivially from the continuity, with respect to λ, of the both sides of Eq. (A.3). □

Consider s square matrices $M^1, \ldots, M^s$ of order d. For any ordered array $(n_1 \cdots n_s)$, set

$$M^n = n_1 M^1 + \cdots n_s M^s.$$

we have the following

Lemma A.4. The determinant of the $d \times d$ matrix M^n is a linear combination of the form

$$\det M^n = \sum_{k_1 \cdots k_d} n_{k_1} \cdots n_{k_d} \det(C_1^{k_1} \cdots C_d^{k_d}),$$

where $k_1, \ldots, k_d$ take values from 1 to s while $C_r^{k_r}$ denotes the r-th column of the matrix M^{k_m}.

The proof is an immediate extension of the well-known result that when all the entries of a column (row) of a square matrix are binomials, its determinant is the sum of the determinants of the two matrices obtained from the original one by replacing, in an orderly way, the binomials with their terms, see, e.g., [9, p. 135].

Consider the $d \times sd$ matrix $\mathcal{M}$ defined as

$$\mathcal{M} = (M^1 \cdots M^s).$$

We have the following.

Theorem A.5. There holds the inequality

$$\operatorname{rank} M^n \leqslant \operatorname{rank} \mathcal{M}.$$

The proof is a straightforward consequence of Lemma A.4.

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Abstract

On adopting the usual view that a second order differential system is hyperbolic provided that the equivalent first order one is, a theorem is proved which offers an intrinsic procedure to check hyperbolicity of second order systems. In deducing such result, the link between second order and first order systems is analyzed in detail and a number of relevant results is proved.

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