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# Higher order shock waves

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## 1. Introduction

In the theory of quasilinear hyperbolic systems, it is a well known result that non-differentiable discontinuous solutions develop from arbitrary initial data; this motivates the introduction of the so-called "weak solutions"—see, e.g., [1, 2]. Explicitly, look at a conservative system of the form

$$\partial_\alpha f^\alpha(U) = f(U), \quad f^0 = U, \tag{1.1}$$

where $\alpha = 0, 1, 2, 3$, $\partial_\alpha = \partial/\partial x^\alpha$, $x^0 = t$, and $f^\alpha, f$ are $N$ dimensional column vectors. If $U(x^\alpha)$ is a piecewise continuous weak solution of (1.1) then the generalized Rankine-Hugoniot conditions

$$[f^\alpha]\partial_\alpha \varphi = 0 \tag{1.2}$$

hold across the surface of discontinuity $\varphi(x^\alpha) = 0$ [3]; in (1.2) the symbol $[\,\cdot\,]$ denotes the jump across the surface of discontinuity. Such a solution is usually called a shock wave.

The purpose of this paper is to show, through an example, that the concept of shock wave could be inadequate for the description of fairly simple physical systems. Specifically, after a brief review in Sect. 2 and some technicalities presented in Sect. 3, we shall calculate the electromagnetic field generated by a charged particle which suffers a jump discontinuity in its velocity $\boldsymbol{v}$ (Sect. 4). As a result, the electromagnetic field not only jumps across a spherical surface expanding at light speed, but also includes a $\delta$-distribution term whose support is the previous surface. This striking situation arises not from crackpot conjectures, but from a hypothesis—$[\boldsymbol{v}] \neq \boldsymbol{0}$—which is usually assumed in gasdynamics [1] and in the theory of electromagnetic pulse, where scattered electrons via Compton effect abruptly pass from rest to a speed $v \neq 0$ and vice versa [4, 5].

The presence of a $\delta$-distribution suggests that we call such a solution a first order shock wave, zero order shock waves being the usual piecewise continuous weak solutions. In general a solution involving the distribution

$\delta^{(n-1)}$, namely the $(n-1)$-th derivative of $\delta$, will be called a $n$-th order shock wave.

Apparently, the Rankine-Hugoniot conditions (1.2) lose their meaning when the field at hand is a higher order shock wave. Thus we need an extended theory capable of embodying solutions of this type. Sect. 5 is devoted to the deduction of the compatibility conditions, for a linear system, which extend Eq. (1.2) to the case of first order shock wave, the extension to higher orders being straightforward.

## 2. Electromagnetic shocks produced by a point charge

So as to discuss electromagnetic shock wave generation by a moving charge, consider a point charge $e$ traveling along the trajectory $\boldsymbol{x} = \boldsymbol{r}(t)$ with velocity $\boldsymbol{v} = \dot{\boldsymbol{r}}$ and adopt the notations

$$\boldsymbol{R}(\boldsymbol{x}, t) = \boldsymbol{x} - \boldsymbol{r}(t), \quad R = |\boldsymbol{R}|, \tag{2.1}$$

$$\varrho = R - \frac{\boldsymbol{R} \cdot \boldsymbol{v}}{c}. \tag{2.2}$$

As well established in the literature—see, e.g., [6]—the point charge generates an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{H}$ which exhibit the familiar splitting into a static and a radiative part according to the relations

$$\boldsymbol{E} = \frac{e}{\varrho^3 \gamma^2}\left(\boldsymbol{R} - \frac{R\boldsymbol{v}}{c}\right) + \frac{e}{c^2 \varrho^3}\boldsymbol{R} \times \left[\left(\boldsymbol{R} - \frac{R\boldsymbol{v}}{c}\right) \times \dot{\boldsymbol{v}}\right], \tag{2.3}$$

$$\boldsymbol{H} = \frac{1}{R}\boldsymbol{R} \times \boldsymbol{E}, \tag{2.4}$$

where, as usual, $\gamma$ stands for the quantity $1/\sqrt{1 - v^2/c^2}$. Of course, the r.h.s. are to be calculated at the retarded time $t'$, the unique solution to the equation

$$t' + \frac{R(\boldsymbol{x}, t')}{c} = t. \tag{2.5}$$

Formulae (2.3), (2.4) permit a description of the electromagnetic field even when the velocity $\boldsymbol{v}$ is a continuous function of $t$ whereas the acceleration $\dot{\boldsymbol{v}}$ is not. Suppose that the charge starts from rest at $\boldsymbol{x} = \boldsymbol{0}$ and that the acceleration takes a constant value $\boldsymbol{a}$ in the time interval $(0, t^*)$ and is zero elsewhere. Then the acceleration suffers jump discontinuities at the instants $t = 0$ and $t = t^*$. Accordingly, the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ jump at the retarded times $t' = 0$ and $t' = t^*$. In this case two shock waves are generated, whose wave fronts can easily be determined by (2.5); it turns out that shock fronts are the spheres

$$|\boldsymbol{x}| = ct, \quad |\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{a}t^{*2}| = c(t - t^*),$$

expanding at light speed $c$. It is a simple matter to specialize Eq. (1.2) to the Maxwell equations—see, e.g., [7]—and to prove that the electromagnetic field (2.3), (2.4) satisfies them whenever $[\dot{\boldsymbol{v}}] \neq \boldsymbol{0}$ and $[\boldsymbol{v}] = \boldsymbol{0}$.

Troubles arise when $t^* \to 0$. In this limiting case the two shock fronts merge together and it is the velocity itself that becomes discontinuous. Then the acceleration is not defined (in the sense of the function theory) and eqs (2.3), (2.4) cannot be used to discuss this problem. To examine properly this case, we have recourse to the distribution theory which allows us to solve the wave equation for the electromagnetic potentials $\Phi$ and $\boldsymbol{A}$ and, from them, to calculate $\boldsymbol{E}$ and $\boldsymbol{H}$. As a result, however, we cannot avoid that a distribution concentrated on the shock front contributes to the electromagnetic field itself.

## 3. Discontinuous solutions to the wave equation

The present task is that of solving the inhomogeneous wave equation for the electromagnetic potentials by allowing the source term to jump in consequence of the discontinuity of the charge velocity. On a more general ground, we first establish the conditions that two solutions of the wave equation, possibly relative to different source terms, can be pieced together.

Mathematically, we suppose that $\psi_0$ and $\psi_1$ are solutions of the differential equations

$$\Box\psi_0 = s_0, \quad \Box\psi_1 = s_1, \tag{3.1}$$

where $s_0, s_1$ denote the source terms and the symbol $\Box$ stands for the D'Alembert operator

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}.$$

Requiring that the two solutions can be cut and pasted together across a surface $\varphi(x^\alpha) = 0$ is tantamount to imposing that the discontinuous function $\psi = \psi_0 + \psi_1 Y(\varphi)$ is a solution of the equation

$$\Box\psi = s_0 + s_1 Y(\varphi), \tag{3.2}$$

$Y$ being the unit step function. On appealing to the formula [8]

$$\frac{\partial Y(\varphi)}{\partial x_\alpha} = \partial_\alpha\varphi Y'(\varphi) = \partial_\alpha\varphi\delta(\varphi),$$

a straightforward calculation shows that

$$\begin{aligned}\Box\psi &= \Box\psi_0 + \Box\psi_1 Y(\varphi)\\ &\quad + \left(2\nabla\psi_1\cdot\nabla\varphi - \frac{2}{c^2}\frac{\partial\psi_1}{\partial t}\frac{\partial\varphi}{\partial t} + \psi_1\Box\varphi\right)\delta(\varphi)\\ &\quad + \psi_1\left[(\nabla\varphi)^2 - \frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2\right]\delta'(\varphi).\end{aligned}$$

In view of (3.1), substitution into (3.2) yields the condition

$$\begin{aligned}&\left(2\nabla\psi_1\cdot\nabla\varphi - \frac{2}{c^2}\frac{\partial\psi_1}{\partial t}\frac{\partial\varphi}{\partial t} + \psi_1\Box\varphi\right)\delta(\varphi)\\ &\quad + \psi_1\left[(\nabla\varphi)^2 - \frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2\right]\delta'(\varphi) = 0.\end{aligned} \tag{3.3}$$

Exploitation of (3.3) relies on the following.

**Lemma.** For every $g(x^\alpha)$ and $h(x^\alpha)$, equation

$$g(x^\alpha)\delta(\varphi) + h(x^\alpha)\delta'(\varphi) = 0$$

holds true if and only if conditions

$$h = 0, \quad g - \frac{\partial h}{\partial\varphi} = 0$$

are verified on the surface $\varphi(x_\alpha) = 0$.

**Proof:** The result is an immediate consequence of the familiar relation $h\delta' = h(0)\delta' - h'(0)\delta$ and of the fact that equation $a\delta + b\delta' = 0$, with $a, b \in \mathbb{R}$, admits the unique solution $a = 0$ and $b = 0$. ■

Consequently, relation (3.3) turns out to be equivalent to the sought conditions on $\psi_1$ and $\varphi$, namely

$$\psi_1\left[(\nabla\varphi)^2 - \frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2\right] = 0, \tag{3.4}$$

$$2\nabla\psi_1\cdot\nabla\varphi - \frac{2}{c^2}\frac{\partial\psi_1}{\partial t}\frac{\partial\varphi}{\partial t} + \psi_1\Box\varphi - \frac{\partial}{\partial\varphi}\left[\psi_1\left((\nabla\varphi)^2 - \frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2\right)\right] = 0, \tag{3.5}$$

which must hold on the surface $\varphi(x^\alpha) = 0$.

Drawing the general implications of (3.4), (3.5) seems to be a formidable task. Here we content ourselves to bring to the reader's attention the following sufficient condition, suggested by physical arguments,

$$(\nabla\varphi)^2 - \frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 = 0, \quad \text{everywhere,} \tag{3.6}$$

which, in turn, makes (3.5) into the form

$$2\nabla\psi_1 \cdot \nabla\varphi - \frac{2}{c^2}\frac{\partial\psi_1}{\partial t}\frac{\partial\varphi}{\partial t} + \psi_1 \Box\varphi = 0, \quad \text{on } \varphi(x^\alpha) = 0. \tag{3.7}$$

This peculiar choice relies on the observation that Eq. (3.6) coincides with the eikonal equation satisfied by every surface traveling at light speed $c$ (see, e.g., [9]); hence such a condition does not seem restrictive at all when dealing with electromagnetic waves in vacuum.

## 4. First order shock waves

Consider the electromagnetic field generated by a charged particle, which starts from rest and abruptly takes the constant velocity $\boldsymbol{v} \neq \boldsymbol{0}$; precisely, suppose that the point charge moves according to the law $\boldsymbol{r}(t) = \boldsymbol{v}tY(t/t_0)$, $t_0$ being a reference time. It seems natural to assume that the relevant electromagnetic potentials $\Phi$ and $\boldsymbol{A}$ can be obtained by piecing together the (retarded) potentials relative to a rest charge with those relative to a charge moving with velocity $\boldsymbol{v}$; in formulae

$$\Phi = \Phi_0 - (\Phi_v - \Phi_0)Y(\varphi), \tag{4.1}$$

$$\boldsymbol{A} = \boldsymbol{A}_v Y(\varphi), \tag{4.2}$$

where the quantities

$$\Phi_0 = \frac{e}{|\boldsymbol{x}|}, \quad \Phi_v = \frac{e}{\varrho}, \tag{4.3}$$

$$\boldsymbol{A}_v = \frac{e\boldsymbol{v}}{c\varrho}, \tag{4.4}$$

are evaluated at the retarded time $t'$, while $\varphi = t'/t_0$.

Of course, the quantities $\Phi_0$, $\Phi_v$, and $\boldsymbol{A}_v$ satisfy the wave equations (3.1) with the appropriate source terms. Accordingly, the potentials (4.1), (4.2) satisfy Eq. (3.2) provided conditions (3.4), (3.5) are verified. Reasoning as in Sect. 2, it turns out that the surface $\varphi = 0$ is the sphere $|\boldsymbol{x}| = ct$ expanding at light speed; hence the sufficient condition (3.6) is automatically satisfied. It is a straightforward application of the relations of Appendix to check that Eq. (3.7) is identically true on the surface $t' = 0$, and that Lorentz condition is fulfilled.

From the familiar relations [6]

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\Phi, \quad \boldsymbol{H} = \nabla \times \boldsymbol{A},$$

and from (4.1) and (4.2) it follows that the electric field takes the form of a first order shock wave, *viz*

$$\boldsymbol{E} = \boldsymbol{E}_0 + (\boldsymbol{E}_v - \boldsymbol{E}_0)Y(\varphi) + \boldsymbol{e}\delta(\varphi), \tag{4.5}$$

the magnetic field being given by Eq. (2.4). The quantities in (4.5) have the following expressions

$$\begin{aligned}
\boldsymbol{E}_0 &= \frac{e\boldsymbol{x}}{|\boldsymbol{x}|^3},\\
\boldsymbol{E}_v &= \frac{e}{\varrho^3\gamma^2}\left(\boldsymbol{R} - \frac{R\boldsymbol{v}}{c}\right),\\
\boldsymbol{e} &= \frac{\varrho\boldsymbol{\varepsilon}}{R} + \frac{e}{ct_0\varrho}\left(\frac{1}{R} - \frac{1}{|\boldsymbol{x}|}\right)\boldsymbol{R},\\
\boldsymbol{\varepsilon} &= \frac{e}{c^2\varrho^3}\left\{\boldsymbol{R} \times \left[\left(\boldsymbol{R} - \frac{R\boldsymbol{v}}{c}\right) \times \frac{\boldsymbol{v}}{t_0}\right]\right\}.
\end{aligned}$$

where $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{v}t'$ in view of (2.1).

As a final remark, we observe that the field $\boldsymbol{E}_v$ coincides with the static part of Eq. (2.3), whereas $\boldsymbol{\varepsilon}$ has the same structure of the radiative part. Moreover, on the surface $\varphi = 0$ we have $\boldsymbol{R} = \boldsymbol{x}$, whence $\boldsymbol{e} = \varrho\boldsymbol{\varepsilon}/|\boldsymbol{x}|$.

## 5. Compatibility conditions

In this Section we determine the compatibility conditions to be satisfied by a first order shock wave across the wave front $\varphi(x^\alpha) = 0$. We restrict ourselves to the case of linear systems, chiefly due to the difficulties of multiplying distributions.

Precisely, consider a linear differential system of the form

$$A^\alpha\partial_\alpha U = CU + B, \tag{5.1}$$

where $A^\alpha$ and $C$ are $N \times N$ matrices, while $B$ is a $N$ dimensional column vector, and look for solutions of the type

$$U = U^{(0)} + (U^{(1)} - U^{(0)})Y(\varphi) + u\delta(\varphi), \tag{5.2}$$

the functions $U^{(0)}$ and $U^{(1)}$ separately being solutions to the system (5.1). In view of this assumption, substitution of (5.2) into (5.1) yield

$$\{A^\alpha((U^{(1)} - U^{(0)})\partial_\alpha\varphi + \partial_\alpha u) - Cu\}\delta(\varphi) + A^\alpha\partial_\alpha\varphi u\delta'(\varphi) = 0.$$

The Lemma provides immediately the sought conditions

$$A^\alpha\partial_\alpha\varphi u = 0, \tag{5.3}$$

$$A^\alpha\{(U^{(1)} - U^{(0)})\partial_\alpha\varphi + \partial_\alpha u\} - Cu - \frac{\partial}{\partial\varphi}(A^\alpha\partial_\alpha\varphi u) = 0, \tag{5.4}$$

which must hold on the surface $\varphi(x^\alpha) = 0$. It should come as no surprise that the electromagnetic field (4.5), (2.4) makes conditions (5.3), (5.4) identically true.

Notice that Eq. (5.3) is similar to the Rankine-Hugoniot conditions (1.2) for linear systems, namely when $f^\alpha = A^\alpha U$; on the contrary, condition (5.4) is typical of first order shock waves. Of course, conditions (5.3), (5.4) reduces to the usual Rankine-Hugoniot equations when $u = 0$. In general, a $n$ order shock wave implies a hierarchy of $n + 1$ compatibility conditions, the first one being of the form (5.3), where $u$ is now the coefficient of $\delta^{(n-1)}$.

## Appendix

Here we collect some useful relations valid when $\boldsymbol{R}(\boldsymbol{x}, t) = \boldsymbol{x} - \boldsymbol{v}t$. A guideline for proving them can be found in [6, p. 186].

*Time derivatives*

$$\frac{\partial \varrho}{\partial t'} = \frac{c\varrho}{R} - \frac{c}{\gamma^2},$$

$$\frac{\partial R}{\partial t'} = \frac{c\varrho}{R} - c, \quad \frac{\partial \boldsymbol{R}}{\partial t'} = -\boldsymbol{v},$$

$$\frac{\partial t'}{\partial t} = \frac{R}{\varrho}, \quad \frac{\partial^2 t'}{\partial t^2} = -\frac{c}{\varrho}\left(-1 + \frac{2R}{\varrho} - \frac{R^2}{\varrho^2\gamma^2}\right).$$

*Spatial derivative at* $t =$ constant

$$\nabla \varrho = \frac{\boldsymbol{R}}{\varrho\gamma^2} - \frac{\boldsymbol{v}}{c},$$

$$\nabla R = \frac{\boldsymbol{R}}{\varrho}, \quad \nabla \otimes \boldsymbol{R} = \boldsymbol{1} + \frac{\boldsymbol{R} \otimes \boldsymbol{v}}{c\varrho},$$

$$\nabla t' = -\frac{\boldsymbol{R}}{c\varrho}, \quad \nabla^2 t' = -\frac{1}{c\varrho}\left(1 + \frac{2R}{\varrho} - \frac{R^2}{\varrho^2\gamma^2}\right),$$

where $\boldsymbol{1}$ denotes the unit tensor.

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**Abstract**

Shock waves and the relevant Rankine-Hugoniot conditions may be inadequate even in simple cases. As an example, the electromagnetic field generated by a point charge, whose velocity jumps from $\boldsymbol{0}$ to a constant value $\boldsymbol{v}$, not only jumps across a spherical surface expanding at light speed, but also includes a $\delta$-distribution term. This suggests that the concept of higher order shock waves be introduced, the associated compatibility conditions being also deduced.

**Sommario**

Il concetto di onda d'urto può rivelarsi inadeguato per descrivere situazioni anche semplici. A titolo di esempio, si determina esplicitamente l'espressione del campo elettromagnetico generato da una carica puntiforme la cui velocità passa istantaneamente da $\boldsymbol{0}$ a un valore costante $\boldsymbol{v}$. Il campo elettromagnetico non solo è discontinuo attraverso una superficie sferica che si espande alla velocità della luce, ma include anche un termine che coinvolge la distribuzione $\delta$. Ciò suggerisce di definire le onde d'urto di ordine superiore, per le quali si determinano le corrispondenti condizioni di compatibilità.

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