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Do Experiments Imply Special Relativity?

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Starting with a thorough and self-contained account of transformations between inertial observers, the most general frame transformation is derived, which fully incorporates the Michelson–Morley experiment and the transverse Doppler effect. Lorentz and Marinov transformations are presented as two particular cases. On a rigorous mathematical ground, the paper presents a theory, more general than special relativity and with three degrees of freedom, that completely agrees with a well-established phenomenology.

1. INTRODUCTION

Since Einstein's celebrated 1905 paper [1], the theory of special relativity has become more and more important; today every sound physical theory assumes, as background, the validity of Einstein's approach. Two arguments strongly support such a theory. The first one is the complete agreement with experiments and the power of the theory in its prevision of new effects. On the other hand, special relativity is a self-consistent mathematical theory with an amazing aesthetic appeal.

In this paper we critically review the experimental bases of special relativity by trying to answer the question whether the most famous experiments, which have contradicted the classical Galilean view, necessarily imply special relativity. With this in mind, we aim at drawing the general mathematical implications which follow from the Michelson–Morley experiment (see, e.g. [2, 3]) and from the transverse Doppler effect [4]. We direct the paper at physicists interested in a rigorous mathematical deduction of the consequences coming from a few natural physical assumptions and from well-established experiments. However, we shall refrain from attaching any particular physical meaning to our mathematical results, because we

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believe that their physical relevance must be proved not by formal arguments, but by firm experimental tests. Devotees of special relativity, as are we, will be content of regarding the paper as an original derivation of the classical Lorentz transformation.

To carry out our program, we first determine the form of the most general frame transformation preserving uniform rectilinear motions. Section 2 shows that such a transformation is a projective one. An obvious physical request forces the transformation to be linear. The general kinematical properties implied by a linear transformation are presented in Section 3. Assumptions about light propagation are stated in Section 4, while Section 5 explicitly calculates the light velocity relative to a generic inertial observer. On the basis of the results so obtained, the most general transformation embodying the Michelson–Morley experiment is determined in Section 6. As a result, four parameters turn out to be undetermined. Appeal to the transverse Doppler effect fixes one of such quantities; mathematically we show that this is equivalent to requiring that the coefficient matrix of the transformation has unitary determinant (Section 7). Lorentz transformation is obtained in Section 8 by imposing that light speed is always c. Nevertheless, a different possibility of choosing the parameters leads to the Marinov transformation [5, p. 33] which allows for absolute simultaneity.

2. The Most General Transformation Preserving Uniform Rectilinear Motions

One of the fundamental blocks to build up classical physics is the assumption that there exists a class of privileged observers—inertial observers—characterized by the following

Principle of mechanical inertia. The motion of isolated mass-points relative to every inertial frame is rectilinear and uniform.

As a first step, we determine the most general transformation among inertial frames, namely the most general transformation making uniform rectilinear motions into uniform rectilinear motions.

Consider two arbitrary inertial observers $\mathscr{F}_0$ and $\mathscr{F}$ and denote by x^α and X^α the space–time coordinates of $\mathscr{F}_0$ and $\mathscr{F}$, respectively. In the following greek indices run from 0 to 3, Latin indices from 1 to 3 and $x^0 = ct$, $X^0 = cT$; in principle the constant c need not be the light speed in vacuum, it suffices that c is a universal constant speed. Assume that space and time coordinates of $\mathscr{F}_0$ and $\mathscr{F}$ are related by the frame transformation

$$X^\alpha = X^\alpha(x^0, x^1, x^2, x^3). \tag{2.1}$$

As shown by Fock in [6, p. 23], the principle of mechanical inertia holds true provided that the transformation (2.1) satisfies the condition

$$\frac{\partial^2 X^\alpha}{\partial x^\beta \partial x^\lambda} = \psi_\beta \frac{\partial X^\alpha}{\partial x^\lambda} + \psi_\lambda \frac{\partial X^\alpha}{\partial x^\beta}, \tag{2.2}$$

where the quantities ψ_β are certain functions of the coordinates x^α.

Indeed the functions ψ_β are fully determined by the system (2.2) itself. Explicitly, multiplication by $\partial x^\beta / \partial X^\alpha$ and summation over α and β yield

$$\psi_\lambda = \frac{1}{5} \frac{\partial^2 X^\alpha}{\partial x^\beta \partial x^\lambda} \frac{\partial x^\beta}{\partial X^\alpha}.$$

The problem of finding the explicit form of transformation (2.1) between two inertial frames is now a purely mathematical one, which consists in solving the differential system (2.2).

THEOREM 2.1. *The most general transformation making uniform rectilinear motions into uniform rectilinear motions is a projective transformation.*

Proof. Consider the integrability conditions for Eq. (2.2)

$$\frac{\partial^3 X^\alpha}{\partial x^\beta \partial x^\lambda \partial x^\mu} = \frac{\partial^3 X^\alpha}{\partial x^\beta \partial x^\mu \partial x^\lambda}. \tag{2.3}$$

From (2.2) calculate explicitly the derivatives, use (2.2) again to get

$$\left(\frac{\partial \psi_\lambda}{\partial x^\mu} - \frac{\partial \psi_\mu}{\partial x^\lambda}\right) \frac{\partial X^\alpha}{\partial x^\beta} - \left(\frac{\partial \psi_\beta}{\partial x^\lambda} - \psi_\beta \psi_\lambda\right) \frac{\partial X^\alpha}{\partial x^\mu} + \left(\frac{\partial \psi_\beta}{\partial x^\mu} - \psi_\beta \psi_\mu\right) \frac{\partial X^\alpha}{\partial x^\lambda} = 0, \tag{2.4}$$

and then multiply (2.4) by $\partial x^\nu / \partial X^\alpha$. By letting first $\nu = \beta$ and subsequently $\nu = \mu$ and performing the implied summations, the integrability conditions (2.3) reduce to

$$\frac{\partial \psi_\alpha}{\partial x^\beta} - \psi_\alpha \psi_\beta = 0, \tag{2.5a}$$

$$\frac{\partial \psi_\alpha}{\partial x^\beta} - \frac{\partial \psi_\beta}{\partial x^\alpha} = 0. \tag{2.5b}$$

Note in passing that (2.5b) is a consequence of (2.5a).

According to (2.5b), there exists a scalar function ψ such that

$$\psi_\alpha = \frac{\partial \psi}{\partial x^\alpha}. \tag{2.6}$$

Inserting (2.6) into (2.5a) and introducing the auxiliary function $\omega = \exp(-\psi)$ allow Eq. (2.5a) to be written as

$$\frac{\partial^2 \omega}{\partial x^\alpha \partial x^\beta} = 0,$$

which implies that $\omega = \Omega + \Gamma_\beta x^\beta$, where Ω and Γ_α are arbitrary constants. Hence $\psi_\alpha = -\Gamma_\alpha/\omega$. Substitution into (2.2) yields the differential conditions

$$\frac{\partial^2(\omega X^\alpha)}{\partial x^\beta\,\partial x^\lambda} = 0;$$

therefore the quantities ωX^α depend linearly on x^β, whence

$$X^\alpha = \frac{A^\alpha + B^\alpha_\beta x^\beta}{\Omega + \Gamma_\beta x^\beta}, \tag{2.7}$$

where A^α and B^α_β are arbitrary constants.

In conclusion the projective transformation (2.7) represents a necessary condition for the validity of the principle of mechanical inertia. That Eq. (2.7) is also sufficient follows from direct calculation. Precisely, the rectilinear and uniform motion

$$x^i = \chi^i + \xi^i t$$

transforms, according to (2.7), into the rectilinear and uniform motion

$$X^i = \frac{M^i N^0 - N^i M^0}{\Sigma N^0 - \Delta M^0} + \frac{\Sigma N^i - \Delta M^i}{\Sigma N^0 - \Delta M^0}\, cT,$$

where

$$M^\alpha = A^\alpha + B^\alpha_j \chi^j, \qquad N^\alpha = B^\alpha_j \xi^j + cB^\alpha_0,$$
$$\Sigma = \Omega + \Gamma_j \chi^j, \qquad \Delta = \Gamma_j \xi^j + c\Gamma_0,$$

thus providing the explicit expressions of the transformed quantities. ∎

Two remarks are in order. First, as is well known [6], adding the constancy of light speed to the principle of mechanical inertia implies necessarily the linearity of transformation (2.1). In full generality Theorem 2.1 shows that leaving aside such a requirement enlarges the class of transformations preserving mechanical inertia. Besides, this amends the assertion that mechanical inertia is equivalent to the linearity of the transformation, as sometime claimed in the literature, see, e.g., [3, p. 7]. Second, it is obvious that Theorem 2.1 is a purely mathematical result. Indeed physics suggests that linearity must be restored through the unavoidable request that the transformation (2.7) be not singular at points having finite coordinates.

3. Properties of a Linear Frame Transformation

In agreement with Theorem 2.1, the last remark of the previous section forces us into assuming the linearity of the frame transformation between the inertial observers $\mathscr{F}_0$ and $\mathscr{F}$. In addition, we lose no generality by considering the homogeneous transformation

$$X^\alpha = \Lambda^\alpha_\beta x^\beta; \tag{3.1}$$

also we remove inessential rotations of the spatial axes by requiring that

$$\lim_{v^i \to 0} \Lambda^\alpha_\beta = \delta^\alpha_\beta, \tag{3.2}$$

v^i standing for the velocity of $\mathscr{F}$ as measured by $\mathscr{F}_0$. For convenience, we split (3.1) into its spatial and temporal parts

$$X^i = S^i_j x^j + W^i t, \tag{3.3a}$$

$$T = H_j x^j + Nt; \tag{3.3b}$$

the splitting (3.3) amounts to writing the matrix Λ^α_β in block notation as

$$\Lambda = \left(\begin{array}{c|c} N & cH \\ \hline W/c & S \end{array} \right).$$

As to the inverse transformation, we put $\Lambda^{-1} = \lambda$ and stipulate that

$$\lambda = \left(\begin{array}{c|c} n & ch \\ \hline w/c & s \end{array} \right);$$

this kind of notation allows us to adopt a duality rule that in any given equation majuscules and minuscules may be interchanged. In the sequel, we shall use freely this rule without any further reference.

We are now interested in qualifying the transformation (3.3) by imposing general physical requirements. To fix terminology, we denote by V^i the velocity of $\mathscr{F}_0$ as measured by $\mathscr{F}$; note carefully that, since we are not assuming any principle of relativity, in general $-V^i$ will be different from v^i, cf. subsequent Eq. (3.7).

As a first requisite, we demand that (3.3) transforms "space into space and time into time preserving time direction"; more formally we assume that

$$\det S \neq 0, \qquad N > 0. \tag{3.4}$$

Note that conditions (3.4) are consistent with the assumptions (3.2).

A further property follows from kinematics. Consider a point P moving with respect to $\mathscr{F}_0$ according to the law $x^i = x^i(t)$; obviously its velocity is

$$u^i = dx^i/dt.$$

In analogy, the velocity of P relative to $\mathscr{F}$ is given by

$$U^i = dX^i/dT.$$

The link between u^i and U^i follows directly from (3.3); the result is

$$U^i = \frac{S^i_j u^j + W^i}{H_j u^j + N}. \tag{3.5}$$

Formula (3.5) allows us to express the velocity V^i in terms of the transformation (3.3), and vice versa. Precisely, since V^i is the velocity of a generic point at rest in $\mathscr{F}_0$, we obtain V^i by setting $u^i = 0$ in (3.5); we have

$$V^i = W^i/N. \tag{3.6}$$

To proceed, note that condition $\Lambda^i_\beta \lambda^\beta_0 = 0$ writes explicitly as $S^i_j w^j + W^i n = 0$. On account of (3.6) and dividing by $n > 0$ we arrive at the remarkable relation

$$S^i_j v^j = -NV^i. \tag{3.7}$$

With the aid of (3.6) and (3.7), it is possible to cast (3.3) and (3.5) into a suggestive form; explicitly, transformation (3.3) becomes

$$X^i = S^i_j (x^j - v^j t), \tag{3.8a}$$

$$T = H_j x^j + Nt, \tag{3.8b}$$

while the addition theorem for velocities (3.5) reads

$$U^i = \frac{S^i_j (u^j - v^j)}{H_j u^j + N}. \tag{3.9}$$

For completeness, we observe that conditions (3.2) reduce to

$$\lim_{v^i \to 0} S^i_j = \delta^i_j, \qquad \lim_{v^i \to 0} H_i = 0, \qquad \lim_{v^i \to 0} N = 1. \tag{3.10}$$

As the previous analysis shows, (3.8) is the most general linear frame transformation endowed with a precise physical meaning. In particular, the simplest expression for (3.8), making (3.10) trivially true, is Galilean transformation. On the other hand, (3.8a) reveals that the general link between inertial observers (including Lorentz transformation) is an anisotropic generalization of Galilean transformation, the matrix S accounting for anisotropy.

4. The Absolute Observer

According to pre-relativistic physics, electromagnetism requires that a distinguished inertial observer is to be chosen in which Maxwell equations take their usual form. We formalize such a requirement through the

Principle of optical inertia. There exists an inertial observer—called the absolute observer—in which light propagates isotropically with speed c.

The crisis of the classical viewpoint arose when experimental devices were set up in order to determine the motion of the Earth with respect to the absolute observer. The most celebrated experiment—the Michelson–Morley experiment—has definitely proved that the classical ideas were to be changed. A possible, and indeed fruitful, way to do this is to follow along the line of thought of special relativity. Here, instead, we interpret the Michelson–Morley result as the proof that light propagates with speed c only when traveling back and forth along any *closed* path. It is our goal to establish the most general form of the frame transformation (3.8) embodying the Michelson–Morley result. Explicitly, we take advantage of the principle of optical inertia by characterizing the link between the absolute observer $\mathscr{A}$ and a generic inertial frame $\mathscr{F}$, where light propagates with speed c only when traveling back and forth along closed circuits.

The procedure just outlined may suggests that the resulting class of transformations between inertial frames is not endowed with a group structure, as physics would require. Fortunately, this is not so. To illustrate this point, denote by $\mathscr{F}_n$ the nth inertial observer and adopt the following notation

$$D_n : \mathscr{A} \to \mathscr{F}_n, \qquad d_n : \mathscr{F}_n \to \mathscr{A} \tag{4.1}$$

for the frame transformation between $\mathscr{A}$ and $\mathscr{F}_n$. Consequently, the transformation T_{nm} from $\mathscr{F}_n$ to $\mathscr{F}_m$ is determined by going from $\mathscr{F}_n$ to $\mathscr{A}$ and then to $\mathscr{F}_m$. In view of (4.1), T_{nm} and its inverse take the form

$$T_{nm} = D_m d_n, \qquad (T_{nm})^{-1} = D_n d_m = T_{mn}, \tag{4.2}$$

while composition satisfies the rule

$$T_{nm} = T_{nr} T_{rm}. \tag{4.3}$$

Since the identity is an allowable transformation as well as any D_n, we conclude that frame transformations constructed according to the above procedure do indeed form a group.

5. The Description of Light Trips

As a basis for discussing Michelson–Morley experiment, we have to evaluate the time it takes a light ray for traveling backward and forward between two points marked on a rigid stick. Also, we have to assume that the stick stays at rest in an arbitrary inertial frame $\mathscr{F}$ and possesses an arbitrary orientation. To reach this goal we should know in advance how light propagates relative to $\mathscr{F}$. This compels us to involve the absolute frame $\mathscr{A}$ in which the physics of light propagation is completely known through the principle of optical inertia. Precisely, light travels isotropically with speed c relative to $\mathscr{A}$, hence the absolute forward and backward velocities are given by

$$u^i_{(f)} = cf^i, \tag{5.1a}$$

$$u^i_{(b)} = cb^i, \tag{5.1b}$$

f^i and b^i being unit vectors. The arbitrariness of the stick orientation amounts to choosing the vector f^i arbitrarily. On the contrary, the vector b^i must be calculated by having recourse to the condition that the observer $\mathscr{F}$ sees light traveling to and fro along the same straight path. Mathematically this implies the existence of a negative constant α such that

$$U^i_{(b)} = \alpha U^i_{(f)}, \tag{5.2}$$

where $U^i_{(f)}$ and $U^i_{(b)}$ denote respectively the forward and backward velocity of the light ray relative to $\mathscr{F}$.

Whereas $U^i_{(f)}$ can be easily calculated by using (3.9) and (5.1a), determining $U^i_{(b)}$ requires the knowledge of the constant α. It is the present task to evaluate such a constant. Let us premise the following

LEMMA 5.1. *The unit vector b^i depends linearly on f^i and v^i according to the formula*

$$b^i = \psi \frac{v^i}{v} + \varphi f^i, \tag{5.3}$$

where $v = (v^i v_i)^{1/2}$ and the constants ψ and φ are given by

$$\psi = \frac{2v(c - v^k f_k)}{c^2 + v^2 - 2cv^k f_k}, \tag{5.4a}$$

$$\varphi = \frac{v^2 - c^2}{c^2 + v^2 - 2cv^k f_k}. \tag{5.4b}$$

Proof. In view of (3.4) the matrix S is non-singular, then application of (3.9) yields

$$(S^{-1})^i_k U^k_{(f)} = \frac{cf^i - v^i}{cH_k f^k + N},$$

$$(S^{-1})^i_k U^k_{(b)} = \frac{cb^i - v^i}{cH_k b^k + N};$$

on account of (5.2) we find

$$\alpha \frac{cf^i - v^i}{cH_k f^k + N} = \frac{cb^i - v^i}{cH_k b^k + N}. \tag{5.5}$$

Equation (5.5) shows that b^i belongs to the linear span of f^i and v^i, hence (5.3) is trivially true. Further exploitation of (5.5) results from its projections on f^i, b^i, and v^i. We obtain the algebraic system

$$\alpha \frac{c - v^k f_k}{cH_k f^k + N} = \frac{cf^k b_k - v^k f_k}{cH_k b^k + N}, \tag{5.6a}$$

$$\alpha \frac{cf^k b_k - v^k b_k}{cH_k f^k + N} = \frac{c - v^k b_k}{cH_k b^k + N}, \tag{5.6b}$$

$$\alpha \frac{cf^k v_k - v^2}{cH_k f^k + N} = \frac{cb^k v_k - v^2}{cH_k b^k + N} \tag{5.6c}$$

in three unknowns $f^k b_k$, $v^k b_k$, and α. Of course, the system (5.6) has the trivial solution $\alpha = 1$ and $b^k = f^k$ which violates the condition $\alpha < 0$ and makes (5.5) into an identity. The relevant solution is determined by dividing (5.6a) and (5.6b) by (5.6c) thus getting rid of α. Straightforward algebra yields

$$f^k b_k = \frac{v^2 - c^2 + 2cv^k f_k - 2(v^k f_k)^2}{c^2 + v^2 - 2cv^k f_k},$$

$$v^k b_k = \frac{2cv^2 - v^k f_k (c^2 + v^2)}{c^2 + v^2 - 2cv^k f_k}.$$

These results and (5.3) provide the values sought for ψ and φ. It is matter of calculation to check that the vector b^i given by (5.3) and (5.4) is in fact a unit vector. ▮

For further reference, we mention here that, as an immediate consequence of (5.4), the constants ψ and φ satisfy the identity

$$\psi = \frac{v}{c}(1 - \varphi). \tag{5.7}$$

The theorem provides us with the explicit expression of the constant α.

THEOREM 5.2. *The non-trivial value of the constant α is*

$$\alpha = \frac{cH_k f^k + N}{cH_k b^k + N} \frac{v^2 - c^2}{c^2 + v^2 - 2cv^k f_k}. \tag{5.8}$$

Proof. Substitution of $f^k b_k$ into (5.6a) yields soon formula (5.8). ▮

It is worth noticing that, in view of (5.4b), Eq. (5.8) takes a more convenient form

$$\alpha = \frac{cH_k f^k + N}{cH_k b^k + N} \varphi. \tag{5.9}$$

The particular choice $H_j = 0$, which includes Galilean transformation too, provides the result $\alpha = \varphi$.

Let us comment upon the condition $\alpha < 0$. It is a peculiarity of (5.8) that condition $v = c$ represents a threshold in order for light to be reflected back to the starting point. Explicitly light reflection occurs when $(cH_k f^k + N)/(cH_k b^k + N) > 0$ and $v < c$ or, alternatively, when $(cH_k f^k + N)/(cH_k b^k + N) < 0$ and $v > c$. It should come as no surprise that Lorentz and Galilean transformations belong to the first case.

6. The Michelson–Morley Experiment Reinterpreted

We are now in a position to formulate, in a fruitful mathematical manner, a reinterpretation of the celebrated Michelson–Morley experiment. As observed by $\mathscr{F}$, consider a light ray which starts from a point P, travels a distance l at a velocity $U^i_{(f)}$, and then, after reflection, gets back to P at a different velocity $U^i_{(b)}$. The lapse of time ΔT between departure and arrival of the light ray is given by

$$\Delta T = \frac{l}{|U_{(f)}|} + \frac{l}{|U_{(b)}|}. \tag{6.1}$$

The very content of the Michelson–Morley experiment is the proof that ΔT takes exactly the value $2l/c$ regardless of the actual values of forward and backward velocities. On this basis, we are led to state the following.

Postulate of constancy of the light speed along closed rectilinear paths. In every inertial frame $\mathscr{F}$, light travels backward and forward along any rectilinear path of arbitrary length l in such a way that relation

$$\frac{l}{|U_{(f)}|} + \frac{l}{|U_{(b)}|} = \frac{2l}{c} \tag{6.2}$$

is satisfied, independently of the orientation of the path.

We stress once more that such an assumption is weaker than the usual principle of constancy of light speed as introduced by Einstein; moreover, no principle of relativity is implied by (6.2). It is worth mentioning that a similar criticism has been raised already in the literature—see, e.g. [7].

The guiding idea is that of looking at condition (6.2) as a restriction on the frame transformation (3.8). Therefore we have to put (6.2) into a more appropriate form. For ease in writing, we introduce two convenient temporary shorthands, viz.,

$$F = cH_k f^k + N, \qquad B = cH_k b^k + N.$$

On account of (5.2) and (5.9), Eq. (6.2) reads

$$|U_{(f)}| = \frac{c}{2}\,\frac{|F|\,|\varphi| + |B|}{|F|\,|\varphi|}. \tag{6.3}$$

The quantity $|U_{(f)}|$ can be calculated with the aid of the addition theorem for velocities (3.9). On introducing the symmetric matrix $\Sigma_{ij} = S_i^k S_{kj}$, squaring (6.3) ultimately yields

$$\Sigma_{jk}(c^2 f^j f^k - 2cf^j v^k + v^j v^k) = \frac{c^2}{4\varphi^2}(F^2\varphi^2 + B^2 + 2|FB\varphi|). \tag{6.4}$$

Now we draw a consequence of the condition $\alpha < 0$, to be imposed in order that light reflects back. In view of (5.8) we find that $FB\varphi = B^2\alpha$ whence $FB\varphi < 0$. Therefore we have

$$(F^2\varphi^2 + B^2 + 2|FB\varphi|) = (F^2\varphi^2 + B^2 - 2FB\varphi) = (F\varphi - B)^2.$$

On account of (5.3), (5.7) and recalling the definitions of F and B, we write (6.4) as

$$\Sigma_{jk}(c^2 f^j f^k - 2cf^j v^k + v^j v^k) = \frac{c^4\psi^2}{4v^2\varphi^2}(H_k v^k + N)^2. \tag{6.5}$$

The quantity $\psi/(v\varphi)$ follows from (5.4). In conclusion, on adopting the usual symbol

$$\gamma = (1 - v^2/c^2)^{-1/2}$$

and introducing the notation

$$\rho = \gamma^2(H_k v^k + N), \tag{6.6}$$

condition (6.4) takes the form

$$(c^2\Sigma_{jk} - \rho^2 v_j v_k) f^j f^k - 2c(\Sigma_{jk} v^k - \rho^2 v_j) f^j + \Sigma_{jk} v^j v^k - \rho^2 c^2 = 0. \tag{6.7}$$

To sum up, Eq. (6.7) is mathematically equivalent to condition (6.2) which guarantees the constancy of the light speed along closed rectilinear paths.

LEMMA 6.1. *Condition* (6.7) *holds for every choice of the unit vector* f^i *if and only if*

$$\Sigma_{ij} = \frac{\rho^2}{\gamma^2}\delta_{ij} + \frac{\rho^2}{c^2} v_i v_j. \tag{6.8}$$

Proof. Condition (6.7) involves the unit vector f^i, which can be chosen arbitrarily; on letting f^i be equal to appropriate unit vectors, condition (6.7) splits up as

$$c^2\Sigma_{ij} - \rho^2 v_i v_j + (\Sigma_{hk} v^h v^k - \rho^2 c^2)\delta_{ij} = 0, \tag{6.9a}$$

$$\Sigma_{ij} v^k - \rho^2 v_i = 0. \tag{6.9b}$$

Multiplication of (6.9a) by $v^i v^j$ provides a relation helpful for subsequent calculations, namely,

$$\Sigma_{ij} v^i v^j - \rho^2 v^2 = 0. \tag{6.10}$$

Indeed use of (6.10) shows that condition (6.9b) is a consequence of (6.9a) and, what is more, makes (6.9a) into the desired result (6.8). This for necessity, sufficiency is obvious. ▌

The last step consists in deducing the explicit form of S_{ij} from (6.8) and from the definition $\Sigma_{ij} = S_i^k S_{kj}$.

THEOREM 6.2. *The most general transformation* (3.8) *between an arbitrary inertial observer and the absolute observer, satisfying the postulate of constancy of the light speed along closed rectilinear paths, is fully determined by the quantities H_i and N, the matrix S_j^i taking on the form*

$$S_j^i = \frac{\rho}{\gamma} \left(\delta_j^i + \frac{\gamma - 1}{v^2} v^i v_j \right). \tag{6.11}$$

Proof. In view of (6.8), we search for a matrix S_{ij} in the form

$$S_{ij} = \frac{\rho}{\gamma} \delta_{ij} + \Xi v_i v_j,$$

Ξ being the quantity to be determined. Substitution into (6.8) yields the values

$$\Xi^{(\pm)} = \frac{\rho}{v^2} \frac{\pm\gamma - 1}{\gamma}$$

which give rise to two different matrices $S^{(\pm)}$. Now $S^{(-)}$ has no limit as $v^i \to 0$; condition (3.10) implies that this solution has to be rejected. Thus we are left with the matrix $S^{(+)}$ which has the expression (6.11), having suppressed the superscript. ▌

An immediate consequence of (6.11) is that the velocities v^i and V^i are parallel. Indeed, as follows from (3.7), (6.11), we have

$$V^i = -\frac{\rho}{N} v^i, \tag{6.12}$$

which represents the sought relationship.

7. Mathematical Analysis of the Transformation

To sum up the results obtained so far, on account of (6.11) the transformation (3.8), between the absolute observer $\mathscr{A}$ and a generic inertial observer $\mathscr{F}$, can be written explicitly as

$$X^i = \gamma(H_k v^k + N)\left(\delta^i_j + \frac{\gamma - 1}{v^2} v^i v_j\right)(x^j - v^j t), \tag{7.1a}$$

$$T = H_k x^k + Nt. \tag{7.1b}$$

The physical significance is clear: formulae (7.1) represent the most general transformation—non-singular at points with finite coordinates and satisfying the physical requirements (3.4)—which is consistent with all the following

(i) principle of mechanical inertia,

(ii) principle of optical inertia,

(iii) postulate of constancy of the light speed along closed rectilinear paths;

also, the transformation (7.1) induces a link between two arbitrary inertial observers through the procedure indicated in Section 4, which, in turn, endows the set of transformations with a group structure.

Apart from the velocity v^i, relative to the absolute observer $\mathscr{A}$, the transformation (7.1) is fully determined by the four quantities H_i and N. Here a subtle point is to be remarked. Given H_i and N, consider the observer $\mathscr{F}$ related to $\mathscr{A}$ by (7.1). Then perform the scale transformation, internal to the observer $\mathscr{F}$,

$$\bar{X}^i = \eta X^i, \qquad \bar{T} = \eta T, \tag{7.2}$$

and assume that the scale factor η is a function of the relative speed v satisfying

$$\lim_{v \to 0} \eta(v) = 1. \tag{7.3}$$

It is readily recognized that the scale transformation (7.2) amounts to making the quantities H_i and N into the corresponding quantities $\bar{H}_i$ and $\bar{N}$ given by

$$\bar{H}_i = \eta H_i, \qquad \bar{N} = \eta N. \tag{7.4}$$

The new quantities (7.4) satisfy (3.10) and hence they single out a possible frame transformation between the same observers $\mathscr{A}$ and $\mathscr{F}$; in this sense the new transformation is similar to the original one. Mathematically, we can define an equivalence relation within the set of frame transformations (7.1) with every equivalence class consisting in those transformations which differ by a scale factor, depending on the relative speed v.

The very problem is to establish what physical requisites are to be imposed so as to single out the relevant scale factor. Several motivations suggest that we fix the scale factor by stipulating that the coefficient matrix of (7.1) has unitary determinant which, in view of (4.2), makes the determinant of every frame transformation unitary between two arbitrary inertial observers. Indeed, Galilean transformation has a unitary determinant. Also, this is the only choice that makes all determinants equal to each other. In so doing, the absolute frame enters each transformation between inertial observers by obeying a request of minimal coupling. Finally, whenever a field theory is to build up on the scenario proposed above, it is natural to require that the action and the Lagrangian density are both invariant. Consequently, the 4-volume element must be invariant, which leads again to the requisite of a unitary determinant.

A definite answer, however, comes from experiments. Specifically, Ives and Stilwell [4] showed experimentally that the transverse Doppler effect is described by the relationship

$$\nu_0 = \frac{\nu}{\gamma}, \tag{7.5}$$

where ν_0 is the frequency of a clock when stationary in the ether, ν its frequency when in motion. According to our notation, this effect is accounted for through the formula

$$\Delta t = n \Delta T, \tag{7.6}$$

which is an immediate consequence of the inverse transformation of (7.1). Incorporating the experimental result (7.5) into our approach is tantamount to requiring that

$$n = \gamma. \tag{7.7}$$

So as to draw the mathematical implication of (7.7), we state the following preliminary

LEMMA 7.1. *In full generality, the relation*

$$n = \frac{1}{N + H_k v^k} \tag{7.8}$$

holds true for the transformation (3.8).

Proof. Note first that condition $\lambda^0_\beta \Lambda^\beta_0 = 1$ implies

$$-h_k S^k_j v^j + nN = 1$$

while condition $\lambda^0_\beta \Lambda^\beta_i = 0$ explicitly reads

$$h_k S^k_i + nH_i = 0.$$

Multiplying the last formula by v^i and substituting into the previous relation yield the sought result. ▮

The mathematical implication of (7.7) is the content of the following

THEOREM 7.2. *The experimental result $n=\gamma$ is mathematically equivalent to the condition that the coefficient matrix Λ of* (7.1) *has a unitary determinant.*

Proof. As a direct calculation shows, the determinant of Λ is given by

$$\det \Lambda = \frac{N\rho^3}{\gamma^2} + \rho H_k v^k, \tag{7.9}$$

while (6.6) and (7.8) imply

$$\rho = \frac{\gamma^2}{n}. \tag{7.10}$$

Assume first that condition $n=\gamma$ holds. Therefore (7.10) becomes $\rho=\gamma$ and (7.9) implies $\det \Lambda = 1$. Suppose now that $\det \Lambda = 1$. Use of (7.9) and (7.10) makes condition $\det \Lambda = 1$ into the formula

$$\frac{\gamma^2}{n}\left(\frac{N\gamma^2}{n^2} + H_k v^k\right) = 1. \tag{7.11}$$

As follows from (7.8), $H_k v^k = 1/n - N$; substitution into (7.11) and some algebraic manipulations provide the equation

$$\left(\frac{\gamma^2}{n^2} - 1\right)\left(\frac{\gamma^2 N}{n} + 1\right) = 0,$$

for the unknown n. Since the second factor is positive and $n > 0$, the only solution is $n=\gamma$. ▮

In accordance with the preceding discussion, we accept the validity of (7.7), which in turn allows the transformation (7.1) to be cast into a more expressive form. Precisely, the general relation (7.8) and condition (7.7) ultimately give

$$X^i = \left(\delta^i_j + \frac{\gamma - 1}{v^2} v^i v_j\right)(x^j - v^j t), \tag{7.12a}$$

$$T = \frac{t}{\gamma} + H_k(x^k - v^k t). \tag{7.12b}$$

Two points must be emphasized. As (7.12a) is identical to the spatial part of the Lorentz transformation, the usual law of contraction of bodies in motion follows also as a consequence of the more general transformation (7.12). Moreover, it is readily recognized that transformation (7.12) is fully determined by knowing the three quantities H_i. Two prominent examples on how choosing H_i are presented in next section.

8. Two Particular Cases

Our concern is now that of deriving the celebrated Lorentz transformation and the Marinov transformation [5] as particular cases of (7.12). Look first at the Lorentz transformation.

According to our view, we are able to arrive at the Lorentz transformation by further imposing that light travels, with respect to $\mathscr{F}$, with a constant speed also along *open* paths. This requisite, which follows as an obvious consequence of the Einstein principle of relativity, amounts to the mathematical condition that

$$|U_{(f)}| = c, \tag{8.1}$$

whereby Eq. (6.2) implies that also $|U_{(b)}| = c$. For ease in calculation, we temporarily adopt the same notations as in (7.1); in the same way we deduced (6.7) from (6.2), we can write Eq. (8.1) in the form

$$(\rho^2 v_j v_k - c^4 H_j H_k) f^j f^k - 2c(\rho^2 v_j + c^2 N H_j) f^j + c^2(\rho^2 - N^2) = 0. \tag{8.2}$$

As Eq. (8.2) must hold for every choice of the unit vector f^i, we recognize that this is true provided that

$$\rho^2 v_i v_j - c^4 H_i H_j + c^2(\rho^2 - N^2)\, \delta_{ij} = 0, \tag{8.3a}$$

$$\rho^2 v_i + c^2 N H_i = 0. \tag{8.3b}$$

Substitution of (8.3b) into (8.3a) yields

$$(\rho^2 - N^2)\left(c^2 \delta_{ij} - \frac{\rho^2}{N^2} v_i v_j\right) = 0,$$

whence

$$\rho^2 = N^2. \tag{8.4}$$

Since Eq. (7.7) and (7.10) imply $\rho = \gamma$, Eq. (8.4) reads $N = \gamma$. Finally, Eq. (8.3b) fully determines H_i in the form

$$H_i = -\frac{\gamma}{c^2} v_i. \tag{8.5}$$

Thus we have arrived at the usual form of the Lorentz transformation.

We turn now our attention to the Marinov transformation. According to

Marinov's view, such a transformation both explains the Michelson–Morley experiment and allows for absolute simultaneity. Here we have to impose only the last request, which is tantamount to setting

$$H_i = 0. \tag{8.7}$$

The interesting point of Marinov transformation is the mathematical possibility of justifying the Michelson–Morley result on a quasi-classical ground without any recourse to Einstein's principle of relativity.

As the previous two examples show, the question put by the title of this paper has definitely a negative answer. However, mathematical arguments cannot help us to make a choice; we leave the question open to the ingenuity of experimental researchers.

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