Journal of Applied Mathematics and Physics (ZAMP)
Vol. 39, September 1988

0044-2275/88/005756-06 $ 2.70/0

Dependence of the wave-front speeds on the propagation direction

By F. Bampi, Dipartimento di Matematica e Applicazioni, Università di Napoli, Via Mezzocannone 8, 80134 Napoli and C. Zordan, Dipartimento di Ingegneria Biofisica ed Elettronica, Università di Genova, Viale F. Causa 13, 16145 Genova, Italy

1. Introduction

A local departure from equilibrium conditions usually gives rise to a propagating perturbation. The study of such a propagation is an effective tool for examining the properties of a given physical system both from a theoretical and experimental point of view. Mathematically it is an attractive feature that a great variety of results can be drawn without any appeal to a specific model. This is so because the behaviour of the major part of physical systems is governed by a set of differential equations which can be cast, ultimately, into the general form of a quasilinear hyperbolic system [1, 2, 3].

According to the general theory, a central role is played by the characteristic speeds c selecting characteristic surfaces across which the normal derivatives of field variables may suffer jump discontinuities. The velocity $\boldsymbol{v} = c\,\boldsymbol{n}$ is called wave velocity of the front moving with normal speed c in a direction $\boldsymbol{n}$; of course c may depend on $\boldsymbol{n}$. The quantity $\partial c/\partial \boldsymbol{n}$ is the velocity with which discontinuities propagate along rays (see e.g. [2]). Remarkably, in the case of linear homogeneous systems (or in the limit of waves of infinite frequency), $\boldsymbol{v}$ is the phase velocity and $\partial c/\partial \boldsymbol{n}$ is the group velocity of sinusoidal waves [4]. Therefore, "a priori" information on the dependence of c on $\boldsymbol{n}$ would be highly profitable so as to gain insights into the structure of the physical system under consideration.

As an outstanding example, note that $\boldsymbol{v}(-\boldsymbol{n})$ differs from $\boldsymbol{v}(\boldsymbol{n})$ according to whether $c(-\boldsymbol{n})$ coincides with $c(\boldsymbol{n})$ or not. In particular, the condition that $c(\boldsymbol{n})$ is homogeneous of degree one, as usually assumed in the literature, implies $\boldsymbol{v}(-\boldsymbol{n}) = \boldsymbol{v}(\boldsymbol{n})$; consequently, propagation along $\boldsymbol{n}$ occurs in one direction only, being forbidden in the opposite direction.

Since that is not the usual case, the concern of this paper is to deduce rigorously the general properties of the function $c(\boldsymbol{n})$. After a brief resumé of wave propagation theory, the main results are presented in Sect. 3. In particular, Theorem 2 provides the general form of the characteristic speeds for first order systems. Second order systems are dealt with in Sect. 4.

2. Preliminaries

Look at a first order quasilinear hyperbolic system, namely

$$u_{,t} + A^k u_{,k} + B = 0 \tag{2.1}$$

where the m-component vector u represents the field variables, depending on time t and on n spatial coordinates $\boldsymbol{x} = (x_1, \ldots, x_n)$; the $m \times m$ matrices A^k $(k = 1, \ldots, n)$ and the m-component vector B depend on u, $\boldsymbol{x}$, and t. A comma stands for partial derivative and the summation convention is assumed throughout. Let S be a moving surface of equation $\phi(\boldsymbol{x}, t) = 0$; denote by $\boldsymbol{n} = \nabla\phi/|\nabla\phi|$ the unit normal to S, and by $c = -\phi,_t/|\nabla\phi|$ the normal speed of propagation. The jumps of the derivatives of u can occur through S provided that c satisfies the characteristic condition

$$\det(n_k A^k - c\,I) = 0 \tag{2.2}$$

where I is the $m \times m$ identity matrix. Owing to hyperbolicity, Eq. (2.2) admits m real (not necessarily distinct) solutions $c^{(\alpha)}$ $(\alpha = 1, \ldots, m)$ for all choices of the unit vector $\boldsymbol{n}$, while the right eigenvectors $d^{(\alpha)}$, satisfying

$$(n_k A^k - c^{(\alpha)} I)\, d^{(\alpha)} = 0, \tag{2.3}$$

constitute a set of m linearly independent vectors. Of course, an analogous property holds also for the left eigenvectors $l^{(\alpha)}$. As a result, the jump of the normal derivative of u through a singular surface moving with a characteristic speed $c^{(\alpha)}$ is an element of the eigenspace belonging to $c^{(\alpha)}$.

3. General form of the characteristic speeds

As follows from (2.2), the characteristic speeds c depend on $\boldsymbol{n}$, besides on u, $\boldsymbol{x}$, and t. Since the aim of this paper is to investigate how c depends on $\boldsymbol{n}$, it is necessary to consider the n components of $\boldsymbol{n}$ as independent variables, thereby removing the request that $\boldsymbol{n}$ is an unit vector. From now on, $\boldsymbol{n}$ denotes an arbitrary vector and $c(\boldsymbol{n})$ a solution to (2.2), the dependence on u, $\boldsymbol{x}$, t being understood. According to this view, a characteristic speed is called isotropic when $c = c(|\boldsymbol{n}|)$.

A preliminary result, on which is based the subsequent analysis, is provided by the following

Lemma. Let $c(\boldsymbol{n})$ be a solution of (2.2). Then

$$c = \frac{\partial c}{\partial \boldsymbol{n}} \cdot \boldsymbol{n}. \tag{3.1}$$

Proof. Differentiate (2.3) with respect to n_k and multiply the result by n_k to obtain

$$\left(n_k A^k - n_k \frac{\partial c}{\partial n_k} I\right) d + (n_p A^p - c\,I)\, n_k \frac{\partial d}{\partial n_k} = 0.$$

Now there always exists a left eigenvector l (belonging to c) such that $l\,d \neq 0$. Multiplication by l cancels out the second term, thus giving

$$l\,(n_k A^k)\, d = \frac{\partial c}{\partial \boldsymbol{n}} \cdot \boldsymbol{n}\,(l\,d).$$

Equation (2.3) implies $l(n_k A^k)\,d = c\,(l\,d)$; hence simplification by $l\,d$ provides the sought result. □

Comparing with the current literature (see e.g. [5]), the peculiar result of this Lemma is that formula (3.1) holds true without any appeal to c being homogeneous of degree one in $\boldsymbol{n}$. Indeed, as it will be shown in the sequel, the homogeneous character of the function $c(\boldsymbol{n})$ is more involved.

To proceed further, the following general result has to be proved.

Theorem 1. Let $f(\boldsymbol{y})$ be a non-vanishing real function and p an integer such that

$$p\, f(\boldsymbol{y}) = \boldsymbol{y} \cdot \nabla f(\boldsymbol{y}). \tag{3.2}$$

Then $f(\boldsymbol{y})$ may be written as

$$\begin{aligned} f(\boldsymbol{y}) &= f_1(\boldsymbol{y}) + f_2(\boldsymbol{y}) && \text{if } p \text{ is odd} \\ f(\boldsymbol{y}) &= f_1(\boldsymbol{y}) && \text{if } p \text{ is even} \end{aligned} \tag{3.3}$$

where, for every real number λ,

$$f_1(\lambda \boldsymbol{y}) = \lambda^p f_1(\boldsymbol{y}) \qquad \text{(homogeneous property)}$$

$$f_2(\lambda \boldsymbol{y}) = |\lambda|^p f_2(\boldsymbol{y}) \qquad \text{(positively homogeneous property)}$$

Proof. Choose $\boldsymbol{y} = \lambda \boldsymbol{v}$, where $\boldsymbol{v}$ is a fixed unit vector. For every $\boldsymbol{v}$ introduce the function $g_v(\lambda) = f(\lambda \boldsymbol{v})$. Of course $\boldsymbol{v} \cdot \nabla f = d g_v / d\lambda$; hence Eq. (3.2) implies

$$p\, g_v(\lambda) = \lambda\, d g_v / d\lambda$$

whence

$$\ln |g_v| = p \ln |\lambda| + k(\boldsymbol{v}).$$

On setting $h(\boldsymbol{v}) = \exp k(\boldsymbol{v})$, the previous formula reads

$$|f(\lambda \boldsymbol{v})| = h(\boldsymbol{v})\, |\lambda|^p.$$

Eliminate the moduli by considering *all continuous solutions* with respect to λ, thus obtaining

$$f(\lambda \boldsymbol{v}) = \lambda^p h(\boldsymbol{v}), \qquad f(\lambda \boldsymbol{v}) = -\lambda^p h(\boldsymbol{v}),$$

$$f(\lambda \boldsymbol{v}) = |\lambda|^p h(\boldsymbol{v}), \qquad f(\lambda \boldsymbol{v}) = -|\lambda|^p h(\boldsymbol{v});$$

the function $h(\boldsymbol{v})$ is now determined by setting $\lambda = 1$. It is then apparent that the first two relations give rise to the same condition

$$f(\lambda \boldsymbol{v}) = \lambda^p f(\boldsymbol{v}), \tag{3.4}$$

whereas the remaining ones lead to

$$f(\lambda \boldsymbol{v}) = |\lambda|^p f(\boldsymbol{v}); \tag{3.5}$$

obviously if p is even Eqs. (3.4) and (3.5) coincide. A simple argument shows that Eqs. (3.4) and (3.5) still hold even if $\boldsymbol{v}$ is replaced by $\boldsymbol{y}$ Accordingly, denote by f_1, f_2 any functions satisfying (3.4), (3.5), respectively. In conclusion, owing to linearity, the most general solution to (3.2) is given by (3.3). □

A consequence of Theorem 1, relevant to the problem at hand, can readily be drawn. Indeed, as shown by the previous Lemma, the characteristic speed $c(\boldsymbol{n})$ satisfies Eq. (3.2) with $p = 1$; hence

Corollary. The characteristic speed $c(\boldsymbol{n})$ may be decomposed as

$$c(\boldsymbol{n}) = c_1(\boldsymbol{n}) + c_2(\boldsymbol{n}),$$

where $c_1(\boldsymbol{n})$ is homogeneous and $c_2(\boldsymbol{n})$ is positively homogeneous of degree one.

It should be pointed out that, unlike $c(\boldsymbol{n})$, the functions c_1 and c_2 are not solutions to (2.2). According to their means, they can be evaluated through the formulas

$$\begin{aligned} c_1(\boldsymbol{n}) &= \tfrac{1}{2}[c(\boldsymbol{n}) - c(-\boldsymbol{n})] \\ c_2(\boldsymbol{n}) &= \tfrac{1}{2}[c(\boldsymbol{n}) + c(-\boldsymbol{n})]. \end{aligned} \tag{3.6}$$

The main result is synthesized in the next Theorem

Theorem 2. The eigenvalues of the matrix $n_k A^k$ are characterized by their homogeneity properties according to the following scheme

(i) pairs of elements of the form $c^{\pm}(\boldsymbol{n}) = c_1(\boldsymbol{n}) \pm c_2(\boldsymbol{n})$
(ii) single elements of the form $c(\boldsymbol{n}) = c_1(\boldsymbol{n})$
(iii) vanishing elements $c(\boldsymbol{n}) = 0$.

Proof. The statements pertinent to (ii) and (iii) are trivial consequences of the previous results. As to statement (i), it is sufficient to show that whenever c^+ is an eigenvalue of $n_k A^k$ then c^- is an eigenvalue too. With this in mind, observe that multiplication of (2.3) by -1 makes the eigenvalue $c(\boldsymbol{n})$ of $n_k A^k$ into the eigenvalue $-c(\boldsymbol{n})$ of $(-n_k)A^k$ and viceversa. Moreover, account of the dependence of c on $\boldsymbol{n}$ implies that $c(-\boldsymbol{n})$ too is an eigenvalue of $(-n_k)A^k$. Hence, if $c(\boldsymbol{n}) = c^+(\boldsymbol{n}) = c_1(\boldsymbol{n}) + c_2(\boldsymbol{n})$ then $(-n_k)A^k$ admits the two eigenvalues $-c^+(\boldsymbol{n})$ and $c^+(-\boldsymbol{n}) = -c_1(\boldsymbol{n}) + c_2(\boldsymbol{n})$, where the homogeneity properties of c_1, c_2 have been used (in passing note that $-c^+(\boldsymbol{n}) \neq c^+(-\boldsymbol{n})$ iff $c_2(\boldsymbol{n}) \neq 0$). Accordingly, the conclusion is that if the matrix $n_k A^k$ has the eigenvalue $c^+(\boldsymbol{n})$, then necessarily it has also the eigenvalue $-c^+(-\boldsymbol{n}) = c^-(\boldsymbol{n})$. □

Observe that whenever c is isotropic, i.e. $c = c(|\boldsymbol{n}|)$, then c is trivially positively homogeneous. Hence, it follows at once from Theorem 2 that $-c$ is a characteristic speed too.

As an example, look at the hyperbolic quasilinear system considered by Donato in Ref. [6]. Such a system has the form (2.1) with $B = 0$ and

$$u = \begin{pmatrix} u_0 \\ u_p \end{pmatrix}; \quad A^k = \begin{pmatrix} a^k & -a^k\alpha_p + b_p^k \\ -\varrho\,\delta_p^k & \varrho\,\alpha_q\,\delta_q^k \end{pmatrix}$$

with $k, p, q = 1, 2, 3$. As shown in [6] the characteristic speeds are given by

$$c^0 = 0 \quad \text{with multiplicity } 2,$$

$$2c^{\pm} = (\boldsymbol{\alpha} + \boldsymbol{a}) \cdot \boldsymbol{n} \pm \{\varrho^2[(\boldsymbol{\alpha} + \boldsymbol{a}) \cdot \boldsymbol{n}]^2 - 4\varrho\,\boldsymbol{n} \cdot \boldsymbol{b}\boldsymbol{n}\}^{1/2}.$$

It turns out that the speeds $c^{\pm}$ are not homogeneous functions while, according to (3.6), the functions

$$\begin{aligned} c_1^{\pm} &= (\boldsymbol{\alpha} + \boldsymbol{a}) \cdot \boldsymbol{n} \\ c_2^{\pm} &= \pm\{\varrho^2[(\boldsymbol{\alpha} + \boldsymbol{a}) \cdot \boldsymbol{n}]^2 - 4\varrho\,\boldsymbol{n} \cdot \boldsymbol{b}\boldsymbol{n}\}^{1/2} \end{aligned}$$

are homogeneous and positively homogeneous respectively, in complete agreement with Theorem 2.

4. Second order conservative hyperbolic systems

Theorem 2 furnishes a general result concerning first order quasilinear hyperbolic systems. However, in many physical examples the governing equations constitute a second order conservative hyperbolic system

$$\frac{\partial}{\partial t} F^\alpha(u^\beta_{,t}, u^\beta_{,h}) + \frac{\partial}{\partial x_k} G^\alpha_k(u^\beta_{,t}, u^\beta_{,h}) = 0\,.$$

As is well known, a suitable choice of the field variables permits the previous system to be cast into the form (2.1). Hence, also in this case, the associated wave-front speeds are completely characterized by Theorem 2. The particular choice $\alpha = \beta = 1$, has been dealt with in Ref. [6]; the form of the characteristic speed is in complete agreement with the present results.

Consider now a special but significant second order system

$$\frac{\partial}{\partial t} F^\alpha(u^\beta_{,t}) + \frac{\partial}{\partial x_k} G^\alpha_k(u^\beta_{,h}) = 0\,; \tag{4.1}$$

as an example, linear elastic anisotropic solids are described by equations of this type [7, 8]. Indeed, (4.1) is a distinguished system because the characteristic speeds are all positively homogeneous. To prove this fact, write (4.1) as

$$\frac{\partial F^\alpha}{\partial u^\beta_{,t}} u^\beta_{,tt} + \frac{\partial G^\alpha_k}{\partial u^\beta_{,h}} u^\beta_{,hk} = 0\,. \tag{4.2}$$

Letting $[\cdot]$ be the jump, applying the compatibility conditions

$$[u_{,tt}] = \xi^\beta c^2; \quad [u^\beta_{,hk}] = \xi^\beta n_h n_k\,,$$

where $\xi^\beta = [n^p n^q u^\beta_{,pq}]$, and setting

$$(A_0)^\alpha_\beta = \frac{\partial F^\alpha}{\partial u^\beta_{,t}}; \quad (A_n)^\alpha_\beta = \frac{\partial G^\alpha_k}{\partial u^\beta_{,h}} n^k n^h\,,$$

the system (4.2) yields

$$[c^2 (A_0)^\alpha_\beta + (A_n)^\alpha_\beta]\,\xi^\beta = 0\,.$$

Non-trivial solutions ξ^β are allowed provided that the quantity $\lambda = c^2$ satisfies the determinantal equation

$$\det\{\lambda (A_0)^\alpha_\beta + (A_n)^\alpha_\beta\} = 0\,.$$

In view of hyperbolicity, all λ's are real and non-negative; hence c is of the form

$$c = \pm\sqrt{\lambda}$$

whereby c turns out to be positively homogeneous.

Acknowledgments

The research leading to this work has been partially supported by the Italian Ministry of Education through the 40% research project "Problemi di evoluzione nei fluidi e nei solidi".

Bibliography

[1] G. Boillat, *La Propagation des Ondes*, Gauthier-Villars, Paris 1965.
[2] A. Jeffrey, *Quasilinear Hyperbolic Systems and Waves*, Pitman, London 1976.

[3] C. Cattaneo, *Elementi di Teoria della Propagazione Ondosa,* Pitagora, Bologna 1976.
[4] M. J. Lighthill, Phil. Trans. Roy. Soc. A *252,* 397–430 (1960).
[5] E. Varley and E. Cumberbatch, *Non-linear theory of wave-front propagation,* J. Inst. Maths. Applics *1,* 101–112 (1965).
[6] A. Donato, *Legge di conservazione supplementare per un particolare sistema iperbolico del primo ordine,* Rend. Acc. Naz. Lincei *56,* 247–252 (1976).
[7] J. D. Achenbach, *Wave Propagation in Elastic Solids,* North Holland, Amsterdam 1975.
[8] E. Dieulesaint and D. Royer, *Elastic Waves in Solids,* Wiley, New York 1980.

Abstract

The dependence of the characteristic speeds of quasilinear hyperbolic systems on the propagation direction $\boldsymbol{n}$ is investigated. It is proved that any non-vanishing characteristic speed $c(\boldsymbol{n})$ is the sum of a homogeneous function $c_1(\boldsymbol{n})$ and a positively homogeneous function $c_2(\boldsymbol{n})$. As a further result, if $c_2(\boldsymbol{n})$ is non-vanishing, then both $c_1(\boldsymbol{n}) \pm c_2(\boldsymbol{n})$ are characteristic speeds.

Sommario

Nel lavoro si analizza la dipendenza dalla direzione di propagazione $\boldsymbol{n}$ delle velocità caratteristiche associate ad un sistema iperbolico quasi lineare. Si prova che ogni velocità caratteristica $c(\boldsymbol{n})$ non nulla è somma di una funzione omogenea $c_1(\boldsymbol{n})$ e di una funzione positivamente omegenea $c_2(\boldsymbol{n})$. Come ulteriore risultato si ha che, se $c_2(\boldsymbol{n})$ è non nulla, allora entrambe le funzioni $c_1(\boldsymbol{n}) \pm c_2(\boldsymbol{n})$ sono velocità caratteristiche.

(Received: December 4, 1987)