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Oblique Incidence of Waves on a Boundary

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With 1 Figure

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Summary

A two-step procedure is developed which allows the emergent modes to be determined when a discontinuity wave strikes obliquely on a boundary. Although the procedure is straightforward, oblique incidence gives rise to a few mathematical problems, which are discussed in detail.

1. Introduction

As it has been recognized since a long time, the mathematical rigour is one of the major advantages presented by the theory of discontinuity wave propagation [1]. Also, the progressive wave propagation, concerning media governed by linear differential equations, is automatically accounted for as a special case [2]. Presently much mathematical effort on wave propagation is developed within this framework.

With this in mind, let us mention the prominent topic regarding the interaction of acceleration waves with strong discontinuities (like shocks or boundaries separating different media). In the literature this problem has been investigated either by linearizing the equations and then considering disturbances of small amplitude (see, e.g., [3], [4]) or, in the particular case of normal incidence, by looking at nonlinear discontinuity waves of arbitrary amplitude (see, e.g., [5], [6]). Indeed, apart from Ref. [7], oblique incidence has been analyzed only in the case of small perturbations (see, e.g., [8]). The point is that a wave-front approach to oblique incidence exhibits unsuspected drawbacks and pitfalls because of several algebraic difficulties which render the analysis really awkward.

The purpose of this paper is to make these problems mathematically precise and to develop a firm basis for future investigations. Thus we set aside the more complicated oblique incidence of waves with shocks by considering as a strong discontinuity a boundary between two different media. In the sequel we restrict ourselves only to plane waves impinging on a plane boundary.

In Sect. 2 we determine admissible emergent modes through a geometric-kinematic procedure leading eventually to Snell's law. In order for the amplitudes of these emergent modes to be calculated, we need a relationship among all the discontinuities of a quantity Q across every wave front, including the boundary (Sect. 3). The final formula resembles Brun's [5], although it is necessary a different interpretation. On the basis of this result, Sect. 4 is devoted to the actual calculation of the amplitudes of the emergent modes in terms of the amplitude of the incident wave. Two cases are examined concerning reflection and refraction or reflection only. Conclusions and comments are presented in Sect. 5.

2. Determination of the Emergent Modes

To make the reflection and refraction pattern unique we must appeal to the requirement of causality [9]. Therefore, we take the incident wave to be the cause whose effect is the generation of the emergent waves; in other words, the emergent waves must propagate away from the boundary. Mathematically, we make this point operative by choosing the unit normal $\boldsymbol{n}$ to each wave front to be directed toward the boundary; accordingly, the incident wave travels with a positive characteristic speed c_{inc}, whereas the characteristic speed c_p of the p-th emergent wave is negative. It is worth noting that if we look at a boundary as a zero-speed strong-discontinuity wave then such a convention formally agrees with the Lax conditions for an evolutionary shock [6], [10].

We determine the emergent modes by a geometric and kinematic analysis about the interaction. Consider the straight line r drawn on the boundary by the incident wave. Of course, the emergent waves must intersect the boundary along the same line r. Definite consequences are obtained by introducing a coordinate system (x, y, z) whereby the boundary is described by the equation $z = 0$ while the unit normal to the incident wave front is $\boldsymbol{n}_{\text{inc}} = (\mu_{\text{inc}}, 0, \sigma_{\text{inc}})$, with $\mu_{\text{inc}}, \sigma_{\text{inc}} > 0$. Then, the incident wave front is given by $\mu_{\text{inc}} x + \sigma_{\text{inc}} z - c_{\text{inc}} t = 0$ and the line r is represented as $x = c_{\text{inc}} t/\mu_{\text{inc}}$. Analogously, the possible p-th outgoing wave front $\mu_p x + \sigma_p z - c_p t = 0$ gives rise to the line $x = c_p t/\mu_p$ on the boundary. The requirement that the two lines coincide leads us to the conclusion

$$\frac{c_{\text{inc}}}{\mu_{\text{inc}}} = \frac{c_p}{\mu_p} \tag{2.1}$$

which, owing to its form, will be referred to as Snell's law.

Evidently, Snell's law allows all the emergent modes to be determined. Indeed, the causality requirement on the outgoing waves provides the signs of the σ_p's whose moduli are given by the obvious formula

$$|\sigma_p| = \sqrt{1 - \mu_p^2}.$$

Hence, the characteristic speeds c_p depend on $\boldsymbol{n}_p$ through μ_p only. Coherently, a characteristic speed c_p selects an admissible emergent mode if Snell's law, regarded as an equation for μ_p, possesses a solution at least. In this case, the emergent mode travels with speed c_p in the direction determined by μ_p.

3. Jump Conditions on a Boundary

Let Q be a quantity, relevant to the body at hand, suffering a strong discontinuity across the boundary in accordance with the generalized Rankine-Hugoniot conditions [6]. In order to find the amplitudes of the emergent modes, we have now to relate all jumps of Q across every wave front with the strong discontinuity of Q across the boundary.

Note first that the line r divides the boundary into two regions b and a, behind and ahead the incident wave front respectively. In agreement with standard notations, denote by Q_p^+ and Q_p^- the (limit) values of Q ahead and behind the p-th wave front; the corresponding jump is defined by $[Q]_p = Q_p^- - Q_p^+$. As to the strong discontinuity, consider as region behind the boundary that containing the incident wave and, accordingly, denote by $[\![Q]\!]_b$, $[\![Q]\!]_a$ the strong discontinuity across the region b and a, respectively. We lose no generality by restricting ourselves to the plane $y = 0$. There, all the wave fronts and the boundary are represented by straight lines intersecting at a point P. The situation is fully depicted in Fig. 1.

So as to find the relationship among the various jumps, consider a circle of radius R centered at P and let all the quantities be calculated on the circle itself. Then, the identity

$$[\![Q]\!]_b - [\![Q]\!]_a - [Q]_p + [Q]_q - [Q]_{\text{inc}} + (Q_q^+ - Q_a^+) + (Q_b^+ - Q_q^-)$$
$$+ (Q_p^- - Q_b^-) + (Q_{\text{inc}}^- - Q_p^+) + (Q_a^- - Q_{\text{inc}}^+) = 0$$

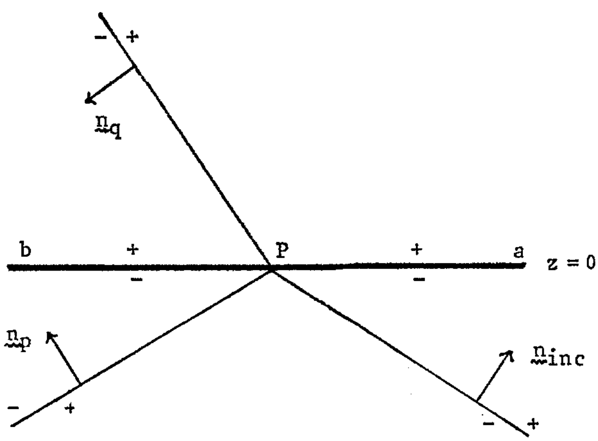

Fig. 1

holds. Taking the limit as R goes to zero the differences enclosed between parentheses go to zero as well. So we are left with the formula

$$[\![Q]\!]_b - [\![Q]\!]_a = [Q]_p - [Q]_q + [Q]_{\mathrm{inc}}.$$

Obviously, if there are M^+ refracted modes and M^- reflected modes, we have

$$[\![Q]\!]_b - [\![Q]\!]_a = \sum_{M^-} [Q]_p - \sum_{M^+} [Q]_q + [Q]_{\mathrm{inc}}. \tag{3.1}$$

In passing we note that a similar formula holds also in the case of normal incidence of shocks and waves [5], [11], although there time limits are involved rather than spatial limits.

4. Amplitudes of the Emergent Modes

Consider first the case when the boundary separates two different media so that reflected and refracted waves may exist. Denote the field variables, describing the behaviour of each medium, by $U_\alpha^\pm$, $\alpha = 1, 2, \ldots$, where the superscripts $\pm$ distinguish between the two media. Finally, suppose that $U_\alpha^\pm$ are the solutions of the following systems of quasilinear hyperbolic differential equations in conservative form

$$\frac{\partial U_\alpha^\pm}{\partial t} + \nabla \cdot F_\alpha^\pm(U_\beta^\pm) = f_\alpha^\pm(U_\beta^\pm). \tag{4.1}$$

Since the boundary acts as a strong discontinuity for the field variables, U_α^+ and U_α^- are connected by the generalized Rankine-Hugoniot relations, which in the present case read

$$F_\alpha^+(U_\beta^+) \cdot \boldsymbol{N} - F_\alpha^-(U_\beta^-) \cdot \boldsymbol{N} = 0, \tag{4.2}$$

where $\boldsymbol{N}$ is the unit normal to the boundary satisfying the condition $\boldsymbol{N} \cdot \boldsymbol{n}_{\mathrm{inc}} > 0$. In order to apply Eq. (3.1), we have to derive Eq. (4.2) with respect to t thus obtaining

$$\frac{\partial F_\alpha^+}{\partial t} \cdot \boldsymbol{N} - \frac{\partial F_\alpha^-}{\partial t} \cdot \boldsymbol{N} = 0$$

which can be re-written as

$$\left[\!\!\left[\frac{\partial F_\alpha}{\partial t} \cdot \boldsymbol{N}\right]\!\!\right] = 0.$$

A straightforward application of (3.1) gives

$$\sum_{M^-} \left[\frac{\partial F_\alpha^-}{\partial t} \cdot \boldsymbol{N}\right]_p - \sum_{M^+} \left[\frac{\partial F_\alpha^+}{\partial t} \cdot \boldsymbol{N}\right]_q + \left[\frac{\partial F_\alpha^-}{\partial t} \cdot \boldsymbol{N}\right]_{\mathrm{inc}} = 0. \tag{4.3}$$

Now the kinematic condition of compatibility relative to a wave front moving with speed c in the direction $\boldsymbol{n}$ reads

$$\left[\frac{\partial F_\alpha}{\partial t}\right] = -c[(\boldsymbol{n} \cdot \nabla) F_\alpha],$$

whence

$$\left[\frac{\partial F_\alpha}{\partial t}\right] \cdot \boldsymbol{N} = -c[(\boldsymbol{n} \cdot \nabla) F_\alpha] \cdot \boldsymbol{N}. \tag{4.4}$$

A more convenient from of (4.4) is obtained as follows. Let $i, j, k = 1, 2, 3$ and adopt the summation convention on repeated indices. Owing to the dependence of F_α on U_β, we have

$$F_{\alpha i,j} = \frac{\partial F_{\alpha i}}{\partial U_\beta} U_{\beta,j}.$$

As usual, we set $\dfrac{\partial F_{\alpha i}}{\partial U_\beta} = (A_i)_{\alpha\beta}$; hence

$$[F_{\alpha i,j}] = (A_i)_{\alpha\beta} [U_{\beta,j}].$$

Maxwell's condition implies that

$$[U_{\beta,j}] = [n_k U_{\beta,k}] n_j.$$

As the theory of wave propagation shows [6], [12], denoting by d_β a right eigenvector of $(A_i)_{\alpha\beta}$ (belonging to the eigenvalue c), we can write

$$[n_k U_{\beta,k}] = \Gamma d_\beta$$

for a suitable amplitude Γ. Collecting the previous results, we obtain

$$[F_{\alpha i,j}] = \Gamma (A_i)_{\alpha\beta} d_\beta n_j.$$

Therefore Eq. (4.4) becomes

$$\left[\frac{\partial F_\alpha}{\partial t}\right] \cdot \boldsymbol{N} = -c\Gamma h_\alpha$$

where $h_\alpha = N_i (A_i)_{\alpha\beta} d_\beta$.

In conclusion, Eq. (4.3) can be cast into the final form

$$\sum_{M^-} (c\Gamma h_\alpha^-)_p - \sum_{M^+} (c\Gamma h_\alpha^+)_q + (c\Gamma h_\alpha^-)_{\text{inc}} = 0. \tag{4.5}$$

After the determination of the emergent modes via Snell's law (2.1), formula (4.5) allows the amplitudes Γ_p, Γ_q ($p = 1, \ldots, M^-$, $q = 1, \ldots, M^+$) to be evaluated algebraically once the amplitude Γ_{inc} is given.

Consider now the special case when a medium is joined with another medium which does not transmit mechanical waves; as an example note that refraction of the elastic waves at an interface of a solid elastic body with air does not occur (or better can be neglected for practical purposes). In this instance, namely when there exists a free surface, the system of waves consists of incident and reflected waves only which can be determined in the following way.

First of all, Snell's law determines the reflected modes. As to their amplitudes, we must appeal to the boundary conditions imposed on the free surface. Let such conditions be of the form

$$B_A(U_\alpha) = 0, \tag{4.6}$$

where the range of A is a subset of the range of α. As the system (4.6) holds identically in time, differentiating with respect to t yields

$$B_{A,\alpha} U_{\alpha,t} = 0$$

whence

$$B_{A,\alpha} [\![U_{\alpha,t}]\!] = 0. \tag{4.7}$$

The validity of the relation

$$[\![U_{\alpha,t}]\!] = -c[n_k U_{\alpha,k}] = -c\Gamma d_\alpha$$

and Eq. (3.1) permit Eq. (4.7) to be written as

$$B_{A,\alpha} \left\{ \sum_{M^-} (c\Gamma d_\alpha^-)_p + (c\Gamma d_\alpha^-)_{\text{inc}} \right\} = 0. \tag{4.8}$$

It is apparent that Eq. (4.8) is less restrictive than Eq. (4.5), in complete agreement with the fact that only the reflected modes are to be determined.

5. Comments

The problem of finding the emergent modes arising from the oblique incidence of an acceleration wave on a strong discontinuity has been solved in two steps. The first one consists in employing Snell's law for determining both the speed and the propagation direction of the admissible emergent modes. In the second step, the amplitudes of the emergent modes are calculated as solutions of the algebraic system (4.5) — or, when only the reflection is involved, of the system (4.8).

The procedure just described looks straightforward. Unfortunately a few drawbacks are present at a theoretical level. On the one hand, the lack of information on the general dependence of c on $\boldsymbol{n}$ (see, however, [13]) does not allow the number of emergent modes to be determined a priori through Snell's law (2.1). On the other, it is a formidable task to evaluate the rank of the algebraic system (4.5). Ultimately, this reflects the difficulty of establishing whether or not the vectors $h_\alpha^\pm$ are linearly independent. Precisely, because of their definition, the number of linearly independent h_α's is not greater than the rank of the matrix $N_i(A_i)_{\alpha\beta}$; this fact may render the system (4.5) underdetermined.

Whereas such an unpleasant feature cannot occur for normal incidence, it is a crucial problem for oblique incidence. We believe that this topic deserves further analysis with a view to deduce a sort of Lax conditions for oblique incidence. This study is underway.

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