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LAGRANGIANS FOR CERTAIN CLASSES OF INELASTIC SOLIDS

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Abstract—The inverse problem of the calculus of variations is investigated in conjunction with two non-linear models of inelastic solids, namely those of Kelvin–Voigt and of Zener. In both cases, on appealing to the general theory, the most general forms of the constitutive equations compatible with the existence of a variational formulation are determined. Next the pertinent (local-in-time) Lagrangian densities are established. The main features of the constitutive equations and the Lagrangian densities so obtained are also discussed.

1. INTRODUCTION

During the last two decades much research has been performed on variational approaches in continuum mechanics[1], this being motivated by theoretical and practical advantages of variational principles against local physical laws. Some of these advantages follow. First, variational formulations of the laws of continuum mechanics may be the only rigorous way to express such laws; this is so because the fundamental principles of continuum mechanics are global in character while local forms of the laws are generated from them only if the involved fields are endowed with suitable smoothness properties. Second, a single functional accounts for all of the intrinsic features of the problem at hand: differential equations, boundary and jump conditions. Third, variational formulations may serve to unify diverse fields and constitute natural means for approximating or finding the solution[2]. Fourth, the variational approach allows a systematic connection between invariance properties and conservation laws[3]. That is why lately particular attention has been addressed to the inverse problem, namely the problem of finding a variational principle corresponding to a given system of differential equations.

In principle, the inverse problem may be answered by appealing to a theorem due to Volterra[4], and revisited by Vainberg[5], which provides a necessary and sufficient condition for an operator to be the derivative of a functional (to be potential). However, so as to make this theorem operative we have to choose a Banach space, in which the operator acts, and a pairing. If, for example, we look at the Banach (and Hilbert) space L_2 and define the pairing as the usual inner product then the functional is characterized by a Lagrangian (density) in the ordinary sense, namely a local-in-time Lagrangian. In such a case systems of non-linear differential equations have been investigated extensively and explicit conditions, ensuring the existence of a variational formulation, have been delivered together with outstanding applications[6–8]. Further variational principles may be achieved by looking at different pairings; in particular, the convolution pairing proved to be very fruitful in several contexts such as, for example, heat conduction[9], viscoelasticity[10, 11], and thermoelasticity[12]. Of course, variational principles involving the convolution pairing are characterized by non-local Lagrangians.

It is often claimed that the existence of variational principles with local-in-time Lagrangians is peculiar to conservative systems. While it is usually so, there seems not to be a general argument ruling out the possibility of variational formulations for dissipative systems. Indeed, it is the aim of this paper to show that some materials which are usually viewed as dissipative bodies, may be cast into a variational framework. Specifically, we look first at Kelvin–Voigt-like materials characterized by the stress tensor being expressed through a non-linear function of the deformation gradient and of the velocity gradient (and also of the space and time variables). Next we examine Zener-like materials[13], namely materials characterized by the time rate of stress tensor being a non-linear function

of the stress tensor itself, of the deformation gradient, and of the velocity gradient. Although the Kelvin–Voigt model might be viewed as a special case, from a variational viewpoint noticeable conceptual aspects make the two problems markedly different; that is why the two models are investigated separately. In both cases we require that the balance equations and the constitutive equations are compatible with the existence of a variational formulation. So we are able to derive significant restrictions on the constitutive functions. For these restricted functions, along with the balance equations, we determine explicitly the corresponding Lagrangian.

If objectivity is accepted as a principle[14] then further restrictions on the constitutive equations are likely to arise; however, for the sake of generality, we disregard objectivity requirements.

2. POTENTIALNESS CONDITIONS AND LAGRANGIANS

Upon labeling the particles of the body under consideration by the positions they occupy in a suitable reference configuration we describe the finite motion of the body through the smooth function $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ determining the position $\mathbf{x}$ of the particle $\mathbf{X}$ at time t. So $\dot{\mathbf{x}} \equiv \mathbf{x}_{,t} = \partial \mathbf{x}(\mathbf{X}, t)/\partial t$ is the velocity and $\mathbf{F} = \partial \mathbf{x}(\mathbf{X}, t)/\partial \mathbf{X}$ is the deformation gradient; we assume that $J \equiv \det \mathbf{F} \neq 0$. Owing to the conservation of mass, the mass density ρ is determined through the motion by

$$\rho = J^{-1}\rho_0(\mathbf{X})$$

where ρ_0 is the reference mass density. In view of this relation it is convenient to account for the stress through the first Piola–Kirchhoff stress tensor $\mathbf{S}$; then the motion is governed by

$$\rho_0 \ddot{\mathbf{x}} = \operatorname{Div} \mathbf{S} + \rho_0 \mathbf{b} \tag{1}$$

where $\mathbf{b} = \mathbf{b}(\mathbf{X}, t)$ is the body force and Div is the material divergence operator. As we are not interested in the thermal aspects of the problem we disregard the temperature field as well as the energy balance equation.

Within such a framework the inverse problem of the calculus of variations consists in ascertaining whether, and how, possible constitutive functions for the stress tensor $\mathbf{S}$ allow the balance eqn (1) to arise from a variational principle. To accomplish this in a systematic way we appeal to the general theory[4, 5, 8]. Now, adopt the summation convention and let a comma followed by a Greek letter, α say, denote partial differentiation with respect to suitable variables y_α, $\alpha = 1, 2, \ldots, n$. According to Ref. [8] a third-order system of the form

$$f_\Gamma(u_\Omega, u_{\Omega,\alpha}, u_{\Omega,\alpha\beta}, u_{\Omega,\alpha\beta\gamma}) = 0, \qquad \Gamma, \Omega = 1, 2, \ldots, m, \qquad \alpha, \beta, \gamma = 1, 2, \ldots, n. \tag{2}$$

in the unknown functions $u_\Omega = u_\Omega(y_\alpha)$, admits a variational formulation if and only if the potentialness conditions

$$\frac{\partial f_\Gamma}{\partial u_{\Omega,\alpha\beta\gamma}} = -\frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha\beta\gamma}} \tag{3}$$

$$\frac{\partial f_\Gamma}{\partial u_{\Omega,\alpha\beta}} = \frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha\beta}} - 3\left(\frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha\beta\gamma}}\right)_{,\gamma} \tag{4}$$

$$\frac{\partial f_\Gamma}{\partial u_{\Omega,\alpha}} = -\frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha}} + 2\left(\frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha\beta}}\right)_{,\beta} - 3\left(\frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha\beta\gamma}}\right)_{,\beta\gamma} \tag{5}$$

$$\frac{\partial f_\Gamma}{\partial u_\Omega} = \frac{\partial f_\Omega}{\partial u_\Gamma} - \left(\frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha}}\right)_{,\alpha} + \left(\frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha\beta}}\right)_{,\alpha\beta} - \left(\frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha\beta\gamma}}\right)_{,\alpha\beta\gamma} \tag{6}$$

are satisfied. Moreover, if conditions (3)–(6) hold then system (2) admits a variational formulation corresponding to the Lagrangian density

$$L(u_\Gamma) = (u_\Omega - \hat{u}_\Omega)\int_0^1 f_\Omega(\tilde{u}_\Gamma, \tilde{u}_{\Gamma,\alpha}, \tilde{u}_{\Gamma,\alpha\beta}, \tilde{u}_{\Gamma,\alpha\beta\gamma})\,\mathrm{d}\xi \tag{7}$$

where $\hat{u}_\Omega$ is a fixed function and $\tilde{u}_\Omega = \hat{u}_\Omega + \xi(u_\Omega - \hat{u}_\Omega)$, $\xi \in [0, 1]$.

Henceforth we use Cartesian tensor notation, the convention that capital indices refer to material coordinates and lower case indices to present coordinates, and the convention that round (square) brackets, enclosing a pair of capital—or lower case—indices, denote symmetrization (skew-symmetrization). Latin indices are allowed to run over 1, 2, 3. Since we aim to apply conditions (3)–(6) to the balance eqn (1) we let $\alpha, \beta, \gamma = 1, 2, 3, 4$; specifically, the independent variables are taken to be $y_M = X_M$, $y_4 = t$. The choice of the unknown functions u_Γ is strictly related to the variational problem under consideration and then it will be made in connection with the specific models for the dissipative body.

3. NON-LINEAR KELVIN–VOIGT MATERIALS

If we allow also for an explicit dependence on $\mathbf{X}$ and t, the non-linear version of the Kelvin–Voigt model is expressed by the response function

$$\mathbf{S} = \mathbf{S}(\mathbf{F}, \dot{\mathbf{F}}, \mathbf{X}, t). \tag{8}$$

Before substituting into the balance eqn (1) and investigating the restrictions placed by conditions (3)–(6), we observe that some care must be exercised when dealing with derivatives involving symmetric quantities; this feature is relevant here because of the dependence of $\mathbf{S}$ on $\dot{\mathbf{F}}$. For example, from $g = g(x_{p,\alpha\beta})$ it follows that

$$\mathrm{d}g = \frac{\partial g}{\partial x_{p,\alpha\beta}}\mathrm{d}x_{p,\alpha\beta} = \frac{\partial g}{\partial x_{p,MQ}}\mathrm{d}x_{p,MQ} + 2\frac{\partial g}{\partial x_{p,Qt}}\mathrm{d}x_{p,Qt} + \frac{\partial g}{\partial x_{p,tt}}\mathrm{d}x_{p,tt}$$

the factor 2 reflecting the symmetry property $x_{p,Qt} = x_{p,tQ}$. Accordingly, in connection with eqn (8) we may write

$$\mathrm{d}S_{iM} = 2\frac{\partial S_{iM}}{\partial x_{p,Qt}}\mathrm{d}x_{p,Qt} + \dots \tag{9}$$

the dots representing terms concerning the other independent variables. Hence we get

$$S_{iM,R} = 2\frac{\partial S_{iM}}{\partial x_{p,Qt}}x_{p,QRt} + \cdots$$

whence

$$S_{iM,M} = 2\frac{\partial S_{i(M}}{\partial x_{p,Q)t}}x_{p,QMt} + \dots .$$

Then balance eqn (1) explicitly reads

$$E_i := \rho_0 x_{i,tt} - \frac{\partial S_{i(M}}{\partial x_{p,Q)}}x_{p,QM} - 2\frac{\partial S_{i(M}}{\partial x_{p,Q)t}}x_{p,QMt} - \frac{\partial S_{iM}}{\partial X_M} - \rho_0 b_i = 0. \tag{10}$$

To ascertain whether eqn (10) may admit a variational formulation we have first to

choose the unknown functions: letting $\Gamma, \Omega = 1, 2, 3$ we set $u_i = x_i$. We observe preliminarly that the term $\rho_0 x_{i,tt}$ does not enter at all the subsequent analysis merely because it gives rise to identities only. Now, as a first step we derive some restrictions placed by conditions (3) and (4). In view of eqn (10) we have

$$\frac{\partial E_i}{\partial x_{p,QMt}} = -\frac{2}{3}\frac{\partial S_{i(M}}{\partial x_{p,Q)t}} \tag{11}$$

the factor 3 being due to the fact that $x_{p,QMt} = x_{p,QtM} = x_{p,tQM}$ whence

$$\mathrm{d}E_i = \frac{\partial E_i}{\partial x_{p,\alpha\beta\gamma}}\mathrm{d}x_{p,\alpha\beta\gamma} + \ldots = 3\frac{\partial E_i}{\partial x_{p,QMt}}\mathrm{d}x_{p,QMt} + \cdots.$$

For ease in calculations it is convenient to make use of the identity

$$\frac{\partial E_i}{\partial x_{p,Qt}} = -\frac{\partial}{\partial x_{p,Qt}}(S_{iM,M}) = -\left(\frac{\partial S_{iM}}{\partial x_{p,Qt}}\right)_{,M}. \tag{12}$$

In view of eqns (11) and (12) we find that

$$\frac{\partial E_i}{\partial x_{p,Qt}} - 3\left(\frac{\partial E_i}{\partial x_{p,QMt}}\right)_{,M} = \left(\frac{\partial S_{iQ}}{\partial x_{p,Mt}}\right)_{,M}.$$

Accordingly, conditions (3) and (4) imply that

$$\frac{\partial S_{i(M}}{\partial x_{p,Q)t}} + \frac{\partial S_{p(M}}{\partial x_{i,Q)t}} = 0 \tag{13}$$

$$\left(\frac{\partial S_{iM}}{\partial x_{p,Qt}} + \frac{\partial S_{pQ}}{\partial x_{i,Mt}}\right)_{,M} = 0. \tag{14}$$

Hence we write

$$\frac{\partial S_{iM}}{\partial x_{p,Qt}} + \frac{\partial S_{pQ}}{\partial x_{i,Mt}} = K_{ipMQ} \tag{15}$$

where $K_{ipMQ} = K_{piQM}$ is a divergence-free tensor function of $\mathbf{X}$ and t. A comparison of eqn (15) with eqn (13) shows that $K_{ip(MQ)} = 0$. Thus we get

$$K_{ipMQ} = K_{[ip][MQ]}$$

and

$$K_{[ip][MQ],M} = 0.$$

As a consequence it follows that there exists a tensor function $\Psi_{ip} = \Psi_{[ip]}$ of $\mathbf{X}, t$ such that

$$K_{ipMQ} = \varepsilon_{MQR}\Psi_{ip,R}.$$

Integration of eqn (15) allows us to find the dependence of $\mathbf{S}$ on $\dot{\mathbf{F}}$. To this purpose we observe that the homogeneous counterpart of eqn (15) may be given the form of a Killing equation (see §84 in Ref. [14]), namely

$$\frac{\partial S_\mu}{\partial \dot{F}_\nu} + \frac{\partial S_\nu}{\partial \dot{F}_\mu} = 0$$

merely by letting the indices μ and ν stand for the pairs iM and pQ, respectively. Then the general solution to eqn (15) is

$$S_{iM} = s_{iM} + A_{iMpQ}\dot{x}_{p,Qt} + \varepsilon_{MQR}\Psi_{ip,R}\dot{x}_{p,Qt} \tag{16}$$

where s_{iM} and $A_{iMPQ} = -A_{pQiM}$ are functions of $\mathbf{F}$, $\mathbf{X}$, and t.

We continue the analysis by exploiting condition (4), in the case $\alpha, \beta = 1, 2, 3$, and condition (5). This is made simpler by having recourse to the identity

$$\frac{\partial E_i}{\partial x_{p,Q}} = -\frac{\partial}{\partial x_{p,Q}} S_{iM,M} = \left(\frac{\partial S_{iM}}{\partial x_{p,Q}}\right)_{,M}$$

and by observing that the right-hand sides of conditions (4) and (5) may be given the form

$$\frac{\partial E_i}{\partial x_{p,QM}} - 3\left(\frac{\partial E_i}{\partial x_{p,QMt}}\right)_{,t} = -\frac{\partial S_{i(M}}{\partial x_{p,Q)}} + 2\left(\frac{\partial S_{i(M}}{\partial x_{p,Q)t}}\right)_{,t} - \frac{\partial E_i}{\partial x_{p,Q}} + 2\left(\frac{\partial E_i}{\partial x_{p,QM}}\right)_{,M}$$

$$+ 2\left(\frac{\partial E_i}{\partial x_{p,Qt}}\right)_{,t} - 6\left(\frac{\partial E_i}{\partial x_{p,QMt}}\right)_{,Mt} = \left[-\frac{\partial S_{iQ}}{\partial x_{p,M}} + 2\left(\frac{\partial S_{iQ}}{\partial x_{p,Mt}}\right)_{,t}\right]_{,M}.$$

As a consequence we see at once that conditions (4) and (5) yield

$$\frac{\partial S_{i(M}}{\partial x_{p,Q)}} - \frac{\partial S_{p(M}}{\partial x_{i,Q)}} + 2\left(\frac{\partial S_{p(M}}{\partial x_{i,Q)t}}\right)_{,t} = 0 \tag{17}$$

$$\frac{\partial S_{iM}}{\partial x_{p,Q}} - \frac{\partial S_{pQ}}{\partial x_{i,M}} + 2\left(\frac{\partial S_{pQ}}{\partial x_{i,Mt}}\right)_{,t} = H_{ipMQ} \tag{18}$$

where H_{ipMQ} ($H_{ipMQ,M} = 0$) is an arbitrary function of $\mathbf{X}$, t; the immediate comparison between eqns (17) and (18) shows that

$$H_{ipMQ} = H_{ip[MQ]}.$$

Since, owing to eqn (16), the dependence of $\mathbf{S}$ on $\dot{\mathbf{F}}$ is known, the integration of eqn (18) allows us to determine the dependence of $\mathbf{S}$ on $\mathbf{F}$. Because of eqn (9) we have

$$2\frac{\partial S_{pQ}}{\partial \dot{x}_{i,Mt}} = A_{pQiM} + \varepsilon_{QMR}\Psi_{pi,R}.$$

Therefore substitution of eqn (16) into eqn (18) gives

$$\left(\frac{\partial A_{iMrR}}{\partial x_{p,Q}} - \frac{\partial A_{pQrR}}{\partial x_{i,M}} + \frac{\partial A_{pQiM}}{\partial x_{r,R}}\right)\dot{x}_{r,Rt} + \frac{\partial s_{iM}}{\partial x_{p,Q}} - \frac{\partial s_{pQ}}{\partial x_{i,M}} + \frac{\partial A_{pQiM}}{\partial t} + \varepsilon_{MQR}\Psi_{ip,Rt} = H_{ipMQ} \tag{19}$$

where the operator $\partial/\partial t$ denotes the derivative with respect to the explicit dependence on t. On interchanging the pairs iM, pQ and adding the resulting equation to eqn (19) we find that

$$\varepsilon_{MQR}\Psi_{ip,Rt} = H_{[ip][MQ]}$$

whence it follows that

$$H_{[ip][MQ],M} = 0$$

holds identically. Then, since $H_{ip[MQ],M} = 0$, a subtraction yields

$$H_{(ip)[MQ],M} = 0.$$

Accordingly there exists a symmetric tensor $\Phi_{ip} = \Phi_{(ip)}$ of $\mathbf{X}$, t such that

$$H_{(ip)[MQ]} = \varepsilon_{MQR}\Phi_{ip,R};$$

the function H_{ipMQ} is thus completely expressed in terms of Ψ_{ip} and Φ_{ip}, namely

$$H_{ipMQ} = H_{ip[MQ]} = \varepsilon_{MQR}(\Phi_{ip,R} + \Psi_{ip,Rt}).$$

Upon substituting this expression into eqn (19) we observe that eqn (19) itself holds identically if and only if

$$\frac{\partial A_{iMrR}}{\partial x_{p,Q}} + \frac{\partial A_{rRpQ}}{\partial x_{i,M}} + \frac{\partial A_{pQiM}}{\partial x_{r,R}} = 0 \tag{20}$$

$$\frac{\partial s_{iM}}{\partial x_{p,Q}} - \frac{\partial s_{pQ}}{\partial x_{i,M}} - \frac{\partial A_{iMpQ}}{\partial t} = \varepsilon_{MQR}\Phi_{ip,R}. \tag{21}$$

The general solution to eqn (20) is

$$A_{iMpQ} = \frac{\partial \Lambda_{iM}}{\partial x_{p,Q}} - \frac{\partial \Lambda_{pQ}}{\partial x_{i,M}}$$

where Λ_{iM} is a function of $\mathbf{F}$, $\mathbf{X}$, t. Consequently eqn (21) becomes

$$\frac{\partial}{\partial x_{p,Q}}\left(s_{iM} - \frac{\partial \Lambda_{iM}}{\partial t}\right) - \frac{\partial}{\partial x_{i,M}}\left(s_{pQ} - \frac{\partial \Lambda_{pQ}}{\partial t}\right) = \varepsilon_{MQR}\Phi_{ip,R}.$$

Thus there exists a scalar function Ξ of $\mathbf{F}$, $\mathbf{X}$, t such that

$$s_{iM} = \frac{\partial \Xi}{\partial x_{i,M}} + \frac{\partial \Lambda_{iM}}{\partial t} + \frac{1}{2}\varepsilon_{MQR}\Phi_{ip,R}x_{p,Q}.$$

In conclusion, we have the expression

$$S_{iM} = \frac{\partial \Xi}{\partial x_{i,M}} + \Lambda_{iM,t} - \frac{\partial \Lambda_{pQ}}{\partial x_{i,M}}x_{p,Qt} + \frac{1}{2}\varepsilon_{MQR}\Phi_{ip,R}x_{p,Q} + \varepsilon_{MQR}\Psi_{ip,R}x_{p,Qt}. \tag{22}$$

It remains now to investigate the restrictions placed by condition (6); upon a straightforward calculation we find that function (22) satisfies identically condition (6).

In summary, the balance eqn (1), with stress $\mathbf{S}$ as given by function (22), admits a variational formulation; indeed, function (22) is the most general constitutive relation compatible with a variational formulation. Additional restrictions on the constitutive equation for $\mathbf{S}$ may arise because of symmetry requirements. Since $\mathbf{S}$ is the (first) Piola–Kirchhoff stress tensor, in the case of non-polar materials the symmetry property

$$S_{iM}x_{j,M} = S_{jM}x_{i,M}$$

must be true. Now, to find the necessary (and sufficient) conditions for this property to hold is a formidable task. It is an easy matter, however, to exhibit an example of a sufficient condition. Let $\Phi_{ip} = 0$, $\Psi_{ip} = 0$ and

$$\Xi = \Xi(\mathbf{C})$$
$$\Lambda_{iM} = x_{i,M}\lambda_{NM}\,\mathrm{tr}\,\mathbf{C}$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, tr denotes the trace, and $\lambda_{NM} = \lambda_{NM}(\mathbf{X}, t)$ is any tensor but the identity. Thus function (22) takes the form

$$S_{iM} = \frac{\partial \Xi}{\partial x_{i,M}} + 2\lambda_{BM}x_{B,M}x_{i,B}x_{p,Q}x_{p,Qt} - 2\lambda_{BQ}x_{i,M}x_{p,B}x_{p,Qt} + \lambda_{NM,t}x_{i,N}\,\mathrm{tr}\,\mathbf{C}$$

satisfying the required symmetry property.

On appealing to expression (7) for the Lagrangian density corresponding to system (2), we move on to determine the Lagrangian density for Kelvin–Voigt materials in the general case, function (22). To simplify the calculations, and without any loss of generality, we put $\hat{u}_i = \hat{x}_i(\mathbf{X})$. Then, letting $\tilde{\mathbf{x}} = \hat{\mathbf{x}} + \xi(\mathbf{x} - \hat{\mathbf{x}})$, expression (7) becomes

$$L(\mathbf{x}) = \frac{1}{2}\rho_0(x_i - \hat{x}_i)x_{i,tt} - \rho_0(x_i - \hat{x}_i)b_i - \int_0^1 (x_i - \hat{x}_i)S_{iM,M}(\tilde{\mathbf{x}})\,\mathrm{d}\xi$$

where the dependence on $\mathbf{X}$, t is understood. In view of function (22) the divergence of $\mathbf{S}$ turns out to be

$$S_{iM,M} = \left(\frac{\partial \Xi}{\partial X_{i,M}}\right)_{,M} + \Lambda_{iM,Mt} - \left(\frac{\partial \Lambda_{pQ}}{\partial x_{i,M}}x_{p,Qt}\right)_{,M}.$$

Because

$$\frac{\partial \Xi}{\partial x_{i,M}}(x_i - \hat{x}_i)_{,M} = \frac{\partial \Xi}{\partial \xi}$$
$$\frac{\partial \Lambda_{pQ}}{\partial x_{i,M}}(x_i - \hat{x}_i)_{,M} = \frac{\partial \Lambda_{pQ}}{\partial \xi}$$
$$(x_i - \hat{x}_i)_{,M} = \frac{\partial \tilde{x}_{i,M}}{\partial \xi}$$

integrations by parts yield

$$\int_0^1 S_{iM,M}(\tilde{\mathbf{x}})(x_i - \hat{x}_i)\,\mathrm{d}\xi = -\int_0^1 \frac{\partial \Xi}{\partial \xi}\,\mathrm{d}\xi - \int_0^1 \frac{\partial}{\partial \xi}[\Lambda_{iM,t}(\tilde{\mathbf{x}})x_{i,M}]\,\mathrm{d}\xi$$
$$+ \int_0^1 \frac{\partial \Lambda_{iM,t}}{\partial \xi}x_{i,M}\,\mathrm{d}\xi + \int_0^1 \frac{\partial \Lambda_{pQ}}{\partial \xi}x_{p,Qt}\,\mathrm{d}\xi + \text{b.t.};$$

the symbol b.t. denotes divergences and time derivatives which result in inessential boundary terms inasmuch as we perform the integration over the space–time variables with the purpose of determining the action of the system. To within a time derivative, the third and fourth integrals appearing on the right-hand side are the opposite of each other. So up to a sign, we obtain the Lagrangian density in the form

$$L(x) = \frac{1}{2}\rho_0 \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} - \Xi(\mathbf{F}) + \Lambda(\mathbf{F}) \cdot \dot{\mathbf{F}} + \rho_0 \mathbf{x} \cdot \mathbf{b} \tag{23}$$

where $\Lambda \cdot \dot{\mathbf{F}} = \Lambda_{iM} x_{i,Mt}$.

Some remarks are in order about the structure, eqn (23), of $L(\mathbf{x})$. First, the functions Ψ_{ip}, Φ_{ip} do not enter the Lagrangian density and the balance equation because they give rise to divergence-free contributions to S. Second, the function Ξ plays the role of a potential energy. This feature should come as no surprise although usually, in continuum mechanics, the prescription $L = T - V$ only works for conservative particle-like systems. Indeed this occurs just because the use of Lagrangian coordinates preserves the similarity with a system of discrete particles. Third, roughly speaking the stress enters the Lagrangian density, eqn (23), as a body force; specifically, to within a time derivative and a divergence, $\rho_0 \mathbf{b} \cdot \mathbf{x} + \Lambda \cdot \dot{\mathbf{F}}$ becomes $(\rho_0 \mathbf{b} + \operatorname{Div} \dot{\Lambda}) \cdot \mathbf{x}$. However, this view must not be overestimated because, as we have seen, the term $(\partial \Lambda_{pQ}/\partial x_{i,M}) x_{p,Qt}$ as well contributes to the body force and to the Lagrangian density.

4. NON-LINEAR ZENER SOLIDS

A non-linear Zener solid is characterized by the differential constitutive equation

$$\tau \dot{\mathbf{S}} = \hat{\mathbf{S}}(\mathbf{S}, \mathbf{F}, \dot{\mathbf{F}}, \mathbf{X}, t) \tag{24}$$

which governs the evolution of stress S; in eqn (24) τ is a parameter with the dimension of time. The presence of $\dot{\mathbf{S}}$ makes the variational problem qualitatively different from that of the previous section. Indeed, unlike the Kelvin–Voigt materials, here we have to deal with a differential constitutive equation—or, alternatively, a functional—; the impossibility of a direct substitution of eqn (24) into eqn (1) compels us to regard stress S as an additional unknown field whose behaviour is determined by eqn (24) which thereby plays the role of the balance equation for S. This view, arising naturally in the variational context, corroborates recent approaches to non-stationary thermodynamics based on kinetic theory[16]. Unfortunately, as a direct application of condition (5) to eqn (24), shows the fact that the highest order derivative, $\dot{\mathbf{S}}$, is an odd-order one rules out the possibility of a variational formulation for the system (1), (24). A remedy to this shortcoming consists in the introduction of an auxiliary tensor function $\boldsymbol{\Sigma}$, defined through the relation

$$\dot{\boldsymbol{\Sigma}} = \mathbf{S}$$

and in looking at $\boldsymbol{\Sigma}$ as an unknown field in place of S. Thus, on introducing the parameter μ simply for dimensional convenience, the full system takes the form

$$\begin{aligned} m_i &:= \mu(\rho_0 x_{i,tt} - \Sigma_{iM,Mt} - \rho_0 b_i) = 0, \\ K_{iM} &:= \tau \Sigma_{iM,tt} - \hat{S}_{iM}(\Sigma_{pQ,t}, x_{p,Q}, x_{p,Qt}, X_Q, t) = 0. \end{aligned} \tag{25}$$

So as to apply conditions (3)–(6) to system (25) we choose as unknown functions u_Γ the fields x_i, Σ_{iM}. Look first at conditions (5) and (6) in connection with eqn $(25)_2$. Since

$$\frac{\partial K_{iM}}{\partial \Sigma_{pQ,t}} = -\frac{\partial \hat{S}_{iM}}{\partial \Sigma_{pQ,t}}$$

condition (5) yields

$$\frac{\partial \hat{S}_{iM}}{\partial \Sigma_{pQ,t}} + \frac{\partial \hat{S}_{pQ}}{\partial \Sigma_{iM,t}} = 0 \tag{26}$$

while condition (6) becomes

$$\left(\frac{\partial \hat{S}_{iM}}{\partial \Sigma_{pQ,t}}_{,t}\right)_{,t} = 0. \tag{27}$$

Observe that eqn (26) may be viewed as a Killing equation; then, on account of eqn (27), the general solution to eqn (26) is

$$\hat{S}_{iM} = \sigma_{iM} + B_{iMpQ}\Sigma_{pQ,t} \tag{28}$$

where $\sigma_{iM} = \sigma_{iM}(\mathbf{F}, \dot{\mathbf{F}}, \mathbf{X}, t)$ and $B_{iMpQ} = -B_{pQiM} = B_{iMpQ}(\mathbf{F}, \dot{\mathbf{F}}, \mathbf{X})$.

Further restrictions on σ_{iM} and B_{iMpQ} follow from the application of conditions (4)–(6) to system (25). To this end observe that

$$\frac{\partial K_{iM}}{\partial x_{p,Qt}} = -\frac{\partial \hat{S}_{iM}}{\partial x_{p,Qt}}, \qquad \frac{\partial m_p}{\partial \Sigma_{iM,Qt}} = -\frac{\mu}{2}\delta_{ip}\delta_{MQ}$$

then, in view of eqn (28), condition (4) reduces to

$$\frac{\partial \sigma_{iM}}{\partial x_{p,Qt}} + \frac{\partial B_{iMrR}}{\partial x_{p,Qt}}\Sigma_{rR,t} = \frac{\mu}{2}\delta_{ip}\delta_{MQ}$$

which holds identically if and only if

$$\frac{\partial B_{iMrR}}{\partial x_{p,Qt}} = 0$$

$$\frac{\partial \sigma_{iM}}{\partial x_{p,Qt}} = \frac{\mu}{2}\delta_{ip}\delta_{MQ}.$$

As a consequence we find that

$$B_{iMPQ} = B_{iMpQ}(\mathbf{F}, \mathbf{X}), \qquad \sigma_{iM} = \mu x_{i,Mt} + \Theta_{iM}(\mathbf{F}, \mathbf{X}, t). \tag{29}$$

On account of system (25) condition (5) reads

$$\frac{\partial \hat{S}_{iM}}{\partial x_{p,Q}} = 0;$$

then, owing to eqns (28) and (29), it turns out that

$$\frac{\partial B_{iMpQ}}{\partial x_{r,R}} = 0, \qquad \frac{\partial \Theta_{iM}}{\partial x_{p,Q}} = 0.$$

In conclusion we arrive at the expression

$$\hat{S}_{iM} = B_{iMpQ}(\mathbf{X})\Sigma_{pQ,t} + \mu x_{i,Mt} + \Theta_{iM}(\mathbf{X}, t) \tag{30}$$

which gives the most general function $\hat{\mathbf{S}}$ making conditions (3)–(6) identically verified; therefore the balance eqn $(25)_1$ and the constitutive eqn $(25)_2$, which now reads

$$\tau \dot{S}_{iM} = B_{iMpQ}(\mathbf{X})\Sigma_{pQ,t} + \mu x_{i,Mt} + \Theta_{iM}(\mathbf{X}, t) \tag{31}$$

admit a variational formulation.

We are now in a position to evaluate explicitly the Lagrangian density corresponding to system (25) and condition (30). In this regard, no generality is lost by choosing the fixed functions $\hat{\mathbf{x}}$ and $\hat{\boldsymbol{\Sigma}}$ as functions of $\mathbf{X}$ only. Then, on letting

$$\tilde{\mathbf{x}} = \hat{\mathbf{x}} + \xi(\mathbf{x} - \hat{\mathbf{x}}), \qquad \tilde{\boldsymbol{\Sigma}} = \hat{\boldsymbol{\Sigma}} + \xi(\boldsymbol{\Sigma} - \hat{\boldsymbol{\Sigma}})$$

eqn (7) becomes

$$L(\mathbf{x}, \boldsymbol{\Sigma}) = \int_0^1 [m_i(\tilde{\mathbf{x}}, \tilde{\boldsymbol{\Sigma}})(x_i - \hat{x}_i) + K_{iM}(\tilde{\mathbf{x}}, \tilde{\boldsymbol{\Sigma}})(\Sigma_{iM} - \hat{\Sigma}_{iM})]\, \mathrm{d}\xi.$$

A straightforward calculation leads to the result

$$L(\mathbf{x}, \boldsymbol{\Sigma}) = \frac{1}{2}\rho_0 \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} + \frac{1}{2}\tau \dot{\boldsymbol{\Sigma}} \cdot \dot{\boldsymbol{\Sigma}} + \mu \mathbf{F} \cdot \dot{\boldsymbol{\Sigma}} + \boldsymbol{\Sigma} \cdot (\mathbf{B}\dot{\boldsymbol{\Sigma}}) + \rho_0 \mathbf{b} \cdot \mathbf{x} + \boldsymbol{\Theta} \cdot \boldsymbol{\Sigma} \tag{32}$$

where $\boldsymbol{\Sigma} \cdot (\mathbf{B}\dot{\boldsymbol{\Sigma}}) = \Sigma_{iM} B_{iMpQ} \dot{\Sigma}_{pQ,t}$.

In ending this section we make a remark on the constitutive eqn (31). The stability properties of eqn (31), which may be viewed as an evolution equation, are mainly due to tensor $\mathbf{B}$[17, 18]. Since $B_{iMpQ} = -B_{pQiM}$, at least one eigenvalue of $\mathbf{B}$ is zero and then the asymptotic stability is not ensured. Specifically, multiplication of eqn (31) by S_{iM} and integration yield, for each particle $\mathbf{X}$

$$S^2(\lambda) - S^2(0) = 2\int_0^\lambda S_{iM}(\mu \dot{F}_{iM} + \Theta_{iM})\, \mathrm{d}t$$

where $S^2 = S_{iM} S_{iM}$. Thus the vanishing of $\dot{\mathbf{F}}$ and $\boldsymbol{\Theta}$ outside a finite interval does not imply that $\mathbf{S}(\lambda) \to \mathbf{0}$ as λ goes to infinity.

5. COMMENTS

In this paper we have concentrated attention on two non-linear models of inelastic continua, namely those of Kelvin–Voigt and Zener. At first sight it seems that the Zener model embraces the Kelvin–Voigt model as a particular case which corresponds to setting $\tau = 0$ whence $\mathbf{S} = \mathbf{S}(\mathbf{F}, \dot{\mathbf{F}}, \mathbf{X}, t)$. However, from a variational standpoint it is not so. Indeed, Sections 3 and 4 show how different the two cases are and how the analysis of the Kelvin–Voigt model is much more complicated. In fact, the Kelvin–Voigt model is described by a third-order differential equation in the unknown function $\mathbf{x}$. The Zener model, instead, ultimately results in a second-order system of differential equations for two unknown functions, namely $\mathbf{x}$ and $\boldsymbol{\Sigma}$.

It is of interest to observe that a preliminary investigation of the Kelvin–Voigt model has been performed in connection with the simplifying assumption that $\mathbf{A}$ be independent of $\mathbf{F}$[19]. If this assumption holds we have $\Lambda_{iM} = \frac{1}{2} A_{iMpQ} x_{p,Q}$ where $A_{iMpQ} = A_{pQiM}$ is a function of $\mathbf{X}$, t. As we should expect, a trivial calculation shows that $\Lambda_{iM,t} - (\partial \Lambda_{pQ}/\partial x_{i,M}) x_{p,Qt}$ reduces to the corresponding term, $\frac{1}{2} A_{iMpQ,t} x_{p,Q} + A_{iMpQ} x_{p,Qt}$, exhibited in Ref. [19].

As a consequence of the analysis so developed, the most general constitutive equation compatible with the existence of a variational formulation is given by function (22) or eqn (31) depending on whether we regard $\mathbf{x}$ or $\mathbf{x}$ and $\boldsymbol{\Sigma}$ as unknown functions. Concerning function (22), since $\mathbf{S}$ is given in terms of four potential functions Ξ, Λ, Φ, Ψ, the model is

likely to be effective in many circumstances. Equation (31), instead, seems not to be applicable to a large extent just because of the skew-symmetry of **B**. As a consequence B_{iMpQ} cannot reduce to the identity tensor $\delta_{ip}\delta_{MQ}$ which corresponds to the Maxwell model. Thus in the one-dimensional case **B** vanishes and then the dissipative character of eqn (31) is ruled out.

In conclusion, we have derived the restrictions on the non-linear Kelvin–Voigt and Zener models ensuring the existence of a variational formulation. These restrictions are rather severe in the case of the Zener model; in particular the Maxwell model cannot admit a variational formulation. Of course these severe restrictions are closely related to the Lagrangian being local-in-time. The literature (see, e.g. Ref. [11]) gives evidence of many possibilities of variational formulations based on convolution functionals (non-local Lagrangians); it is our intention to re-examine models of dissipative materials and to derive the conditions allowing them to arise from non-local Lagrangians. In this connection, looking at the inverse problem in the extended sense[20] could be highly profitable.

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