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Third-Order Tensor Potentials for the Riemann and Weyl Tensors. II: Singular Solutions

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Abstract

The analysis of the admissibility of a potential representation for the Riemann tensor is here continued. As in the preceding paper, the starting point is to regard the relationship between the Riemann tensor and its possible potential as a system of partial differential equations determining the unknown potential. The first result, strengthening a previous conclusion, is that there never exist ordinary solutions. Surprisingly enough, in a four-dimensional Riemannian manifold the existence of singular solutions is established without requiring any integrability condition. Possible applications and generalizations are also suggested.

§(1): *Introduction*

In a previous paper [1] (henceforth denoted by I) we examined the possibility of expressing the Riemann and Weyl tensors in terms of the covariant derivatives of third-order tensor potentials [2, 3]. There we emphasized the role of such representations within the framework of relativity and geometry, also through specific examples. However it was the central point of our analysis to decide whether such representations are allowed or not by regarding them as partial differential equations in the unknown tensor potentials. On appealing to Cartan's theory [4], we were able to conclude that the Weyl tensor always admits Lanczos' representation [2], whereas we found that the corresponding Brinis-

Udeschini's formula [3] for the Riemann tensor, namely,

$$R_{abcd} = H_{abc;d} - H_{abd;c} + H_{cda;b} - H_{cdb;a} \tag{1}$$

where $H_{abc} = H_{[ab]c}, H_{[abc]} = 0$, does not always admit a solution H_{abc} for a given R_{abcd}. Since our aim is to proceed with the investigation of equation (1), we must illustrate, in a more rigorous way, the pertinent results obtained in I.

So as to avoid algebraic troubles due to the cyclic property of R_{abcd}, we showed that the existence of a solution to equation (1) is mathematically equivalent to the existence of a solution $T_{abc} = T_{[ab]c}$ to equation

$$N_{abcd} = T_{\{abc;d\}} \tag{2}$$

where N_{abcd} enjoys the symmetry properties

$$N_{abcd} = N_{[ab][cd]} = N_{cdab} \tag{3}$$

In (2) the curly braces are a shorthand notation for the following linear operation on the indices $(abcd)$:

$$\{(abcd)\} = \tfrac{1}{8}\,[(abcd) - (bacd) - (abdc) + (badc) + (cdab) - (cdba) - (dcab) + (dcba)]$$

where the factor $\frac{1}{8}$ is introduced here for making the braces into a projection operator. Besides this reformulation of the problem (1), we introduced the hypothesis of generic conditions by letting the data N_{abcd} be constrained by (3) only. As a consequence of this definition, the field N_{abcd} is generic if the number of its nonnull algebraically independent components is exactly 21, that is the maximum number compatible with (3). Therefore we exhibited the proof that, under generic conditions on N_{abcd}, equation (2) does not admit any ordinary[1] analytic solution. It is important to realize that such a result does not prevent the existence of nonordinary solutions to (2) and, in turn, to (1) (cf. examples presented in I).

This paper is devoted to making the last statement precise and operative. To this end, the first unavoidable step consists in a preliminary analysis of the conditions for an ordinary solution to equation (2) to exist; see I, equation (23). The analysis shows that the hypothesis of generic conditions can be removed and, meanwhile, leads to the conclusion that equation (2) never admits ordinary solutions (Section 2). This result enables us to look at singular solutions by

[1] Sometimes, as we did in I, ordinary solutions are called regular solutions; here we prefer the use of ordinary solutions in order to adhere more closely to Cartan's terminology [4]. Observe in passing that Cartan's procedure provides also a self-consistent and nonambiguous classification scheme for solutions to systems of partial differential equations, which we shall refer to in the sequel. Roughly speaking, ordinary solutions and nonordinary solutions (or rather Cartan's singular solutions) correspond, respectively, to the usual general solutions and the usual singular solutions to a system of partial differential equations (for technical details on this remark see, e.g., [4, 5]).

having recourse to Cartan's prolongation procedure [4]. The surprising feature of the problem at hand is that, in the four-dimensional case, equation (2) does always admit singular solutions irrespective of the data N_{abcd} and of the underlying geometry; in other words, the existence of singular solutions to (2) is not subject to any integrability condition (Section 3). In Section 4 we briefly summarize the content of the paper with a view to pointing out posssible applications and generalizations of the representation (2). In particular we mention that our results still hold even if the Levi-Civita connection is replaced by an arbitrary affine connection. As a final remark, we also discuss the existence of singular solutions to (2) in an n-dimensional Riemannian manifold, $n \neq 4$.

Following I, the paper has been so organized that the proofs of the theorems may be omitted without loss of mathematical continuity. Nevertheless we would like to stress that the technical aspects of our approach to (1) become important insofar as they constitute a nontrivial application of Cartan's systematic procedure for the study of partial differential equations. Within this context, the present problem provides also an example where the nonexistence of ordinary solutions does not imply any integrability condition for the existence of singular solutions.

§(2): *Nonexistence of Ordinary Analytic Solutions*

The application of Cartan's geometric theory of partial differential equations relies upon a preliminary reformulation of the problem at hand in terms of exterior differential equations. In our case, as shown in detail in I, equation (2) is equivalent to the closed exterior differential system

$$Z_{\{abcd\}} - T_{e\{ad}\Gamma^{e}_{bc\}} + T_{e\{ac}\Gamma^{e}_{bd\}} - N_{abcd} = 0 \tag{4a}$$

$$dT_{abc} - Z_{abce}\, dx^{e} = 0 \tag{4b}$$

$$dZ_{\{abcd\}} - A_{abcde}\, dx^{e} = 0 \tag{4c}$$

$$dZ_{abce} \wedge dx^{e} = 0 \tag{4d}$$

where

$$A_{abcde} = A_{\{abcd\}e} = Z_{f\{ad|e|}\Gamma^{f}_{bc\}} - Z_{f\{ac|e|}\Gamma^{f}_{bd\}} + T_{f\{ad}\Gamma^{f}_{bc\},e} - T_{f\{ac}\Gamma^{f}_{bd\},e} + N_{abcd,e} \tag{5}$$

The system (4) is defined on a formal 124-dimensional (analytic) manifold $\mathfrak{M}$, whose local coordinates are $(x^{a}, T_{abc}, Z_{abcd})$. So as to arrive at integrability conditions for the system (4) through Cartan's geometric procedure, the nonexistence result of I must necessarily be refined. Precisely, the generic conditions on N_{abcd} must be removed thereby obtaining the proper mathematical setting for making Cartan's procedure operative.

The following theorem, holding for every field N_{abcd}, generalizes the statement (i) of the theorem proved in I and hence represents the crucial starting point for the subsequent investigation.

Theorem 1. Consider an analytic four-dimensional Riemannian manifold and let N_{abcd} be an analytic tensor field satisfying the symmetry properties (3). Then there does not exist any local ordinary analytic solution to equation (2) whatever be the choice of N_{abcd}.

Proof. The theorem is proved by showing that the exterior system (4) on $\mathfrak{M}$ does not admit any ordinary four-dimensional integral manifold involutive with respect to $x^1, \ldots, x^4$. Accordingly, we have recourse to Cartan's necessary and sufficient condition for involutiveness (see [4], p. 91) which requires that the reduced Cartan characters s_i' coincide with the corresponding Cartan characters s_i. The integers s_i' and s_i are evaluated through a step-by-step procedure, described in detail in I, which will be used here without any further reference.

Let P be a point of $\mathfrak{M}$ where (4a) holds. Introduce the unknown vector

$$X = X^a \frac{\partial}{\partial x^a} + X_{abc} \frac{\partial}{\partial T_{abc}} + X_{abcd} \frac{\partial}{\partial Z_{abcd}}$$

belonging to the tangent space to $\mathfrak{M}$ at P and consider the algebraic linear system

$$X_{abc} - Z_{abce} X^e = 0 \tag{6a}$$

$$X_{\{abcd\}} - A_{abcde} X^e = 0 \tag{6b}$$

The first step consists in determining the vector $Y_{(1)}$ as the general solution to (6). To this end, we observe that any tensor $F_{abcd} = F_{[ab]cd}$ may uniquely be split as

$$F_{abcd} = \tilde{F}_{abcd} + \bar{F}_{abcd} + \hat{F}_{abcd}$$

where

$$\tilde{F}_{abcd} = \tilde{F}_{\{abcd\}}$$
$$\bar{F}_{abcd} = \bar{F}_{ab[cd]} = -\bar{F}_{cdab}$$
$$\hat{F}_{abcd} = \hat{F}_{ab(cd)}$$

Since $A_{abcde} = A_{\{abcd\}e}$, we find that

$$Y^{(1)}_{abc} = Z_{abcd} Y^d_{(1)} =: Z_{abc1}$$
$$Y^{(1)}_{abcd} = A_{abcd1} + \bar{Y}^{(1)}_{abcd} + \hat{Y}^{(1)}_{abcd} \tag{7}$$

where $Y^d_{(1)}, \bar{Y}^{(1)}_{abcd}, \hat{Y}^{(1)}_{abcd}$ are arbitrary tensors satisfying the pertinent symmetries. The notation $V_{\cdot\cdot\alpha} = V_{\cdot\cdot a} Y^a_{(\alpha)}$ will be used throughout.

In the second step we calculate the vector $Y_{(2)}$, solution to (6) and to

$$X_{abcd}Y^d_{(\alpha)} - Y^{(\alpha)}_{abcd}X^d = 0 \tag{8}$$

where $\alpha = 1$. It turns out that

$$\begin{aligned} Y^{(2)}_{abc} &= Z_{abc2} \\ Y^{(2)}_{abcd} &= A_{abcd2} + \bar{Y}^{(2)}_{abcd} + \hat{Y}^{(2)}_{abcd} \end{aligned} \tag{9}$$

where $Y^a_{(2)}$ is arbitrary and $Y^{[a}_{(1)}Y^{b]}_{(2)} \neq 0$, while $\bar{Y}^{(2)}_{abcd}$ and $\hat{Y}^{(2)}_{abcd}$ satisfy the further relation

$$\bar{Y}^{(2)}_{abc1} + \hat{Y}^{(2)}_{abc1} = \bar{Y}^{(1)}_{abc2} + \hat{Y}^{(1)}_{abc2} + A_{abc21} - A_{abc12} \tag{10}$$

as a consequence of (7), (8), (9).

As shown in I, it is the third step that gives rise to the first internal identity in the evaluation of s_2', whereas in the corresponding evaluation of s_2 it leads to the condition

$$Y^{(2)}_{121e} - Y^{(1)}_{122e} - 2A_{1212e} = 0$$

which must be identically true for the system (4) to be involutive. Now, in view of (7), (9), (10), the previous condition reads

$$\bar{Y}^{(1)}_{12e2} - \bar{Y}^{(2)}_{12e1} - 3A_{12[12e]} = 0 \tag{11}$$

which implies significant restrictions on $\bar{Y}^{(1)}_{abcd}$ and on $\bar{Y}^{(2)}_{abcd}$. Since Cartan's procedure does not allow the imposition of further restrictions on the solutions to (6), (8), we conclude that condition (11) can never be satisfied. Hence the system (4) is not involutive with respect $x^1, \ldots, x^4$, whatever be the data N_{abcd}. ■

An intuitive idea of the content of Theorem 1 may be obtained as follows. By construction, the system (4) is closed in $\mathfrak{M}$ and therefore it is completely integrable in $\mathfrak{M}$; then there exists the class of the four-dimensional ordinary integral manifolds belonging to the *general* solution of the system (4). What Theorem 1 tells us is that the equations defining the generic four-dimensional integral element always imply relationships among $dx^1, \ldots, dx^4$. Hence we cannot take $x^1, \ldots, x^4$ as independent variables for the system (4), in the case of four-dimensional integral manifolds. Accordingly, possible solutions to (2) are necessarily related to *singular* solutions to (4).

§(3): *Existence of Singular Analytic Solutions*

The method for obtaining singular solutions to an exterior differential system with p independent variables has been introduced in the literature by E. Cartan through the notion of prolongation. In essence, this consists in a systematic procedure based on the addition of new equations, and possibly of new variables, to the original system in such a way that its solutions are in one-to-one correspondence with the solutions of the newly constructed system. The interest in this prolongation procedure is that, under certain conditions, every solution to the original system becomes a general solution after a finite number of prolongations [6] (see also [5]).

On appealing to this technique, we are able to prove the following.

Theorem 2. Consider an analytic four-dimensional Riemannian manifold and let N_{abcd} be an analytic tensor field satisfying the symmetry property (3). Then there do always exist local singular analytic solutions to equation (2).

Proof. In the present case, Cartan's prolongation procedure goes as follows. First of all, complete the set of 1-forms appearing in (4) to a base of the cotangent space to $\mathfrak{M}$ at P by introducing the 1-forms $\omega_{abcd} = dZ_{abcd} - dZ_{\{abcd\}}$. Then express ω_{abcd} in terms of $dx^1, \ldots, dx^4$; in full generality one has

$$\omega_{abcd} = (\bar{W}_{abcde} + \hat{W}_{abcde})\, dx^e \tag{12}$$

where $\bar{W}_{abcde} = \bar{W}_{[ab][cd]e} = -\bar{W}_{cdabe}$, $\hat{W}_{abcde} = \hat{W}_{[ab](cd)e}$. Now, because of (4c), substitution of (12) into (4d) yields

$$\bar{W}_{abc[de]} + \hat{W}_{abc[de]} + A_{abc[de]} = 0 \tag{13}$$

As a consequence of the last paragraph of the Appendix, in the four-dimensional case equations (5) are always algebraically compatible independently of the choice of the data A_{abcde}. Hence the first prolongation of the system (4) is achieved by regarding $\bar{W}_{abcde}$ and $\hat{W}_{abcde}$ as new unknowns and by adding to equations (4) the new equations (12), (13) and their exterior derivatives. In so doing the prolonged system turns out to be defined on a formal 424-dimensional (analytic) manifold $\mathfrak{N}$ whose local coordinates are $(x^a, T_{abc}, Z_{abcd}, \bar{W}_{abcde}, \hat{W}_{abcde})$. Explicitly the prolongation reads

$$Z_{\{abcd\}} - T_{e\{ad}\Gamma^e_{bc\}} + T_{e\{ac}\Gamma^e_{bd\}} - N_{abcd} = 0 \tag{14a}$$

$$\bar{W}_{abc[de]} + \hat{W}_{abc[de]} + A_{abc[de]} = 0 \tag{14b}$$

$$dT_{abc} - Z_{abce}\, dx^e = 0 \tag{14c}$$

$$dZ_{abcd} - (\bar{W}_{abcde} + \hat{W}_{abcde} + A_{abcde})\, dx^e = 0 \tag{14d}$$

$$d\bar{W}_{abc[de]} + d\hat{W}_{abc[de]} + B_{abc[de]f}\, dx^f = 0 \tag{14e}$$

$$d\bar{W}_{abcde} \wedge dx^e = 0 \tag{14f}$$

$$d\hat{W}_{abcde} \wedge dx^e = 0 \tag{14g}$$

where $B_{abcdef} = B_{\{abcd\}ef}$ is defined by $dA_{abcde} = B_{abcdef}\, dx^f$. It should be noted that, since equation (4c) is closed, its exterior derivative is identically zero. Hence $dA_{abcde} \wedge dx^e$ vanishes identically, whence

$$B_{abcdef} = B_{abcd(ef)} \tag{15}$$

We are now in a position to prove that the system (14) is involutive with respect to $x^1, \ldots, x^4$ by showing that $s_i' = s_i$ $(i = 0, \ldots, 3)$. From now on, the procedure follows along the lines of Theorem 1 (see also I).

Let Q be a point of $\mathfrak{N}$ where (14a, b) hold. The characters s_0' and s_0 are related to the rank of the algebraic linear system

$$X_{abc} - Z_{abcd}X^d = 0 \tag{16a}$$

$$X_{abcd} - (\bar{W}_{abcde} + \hat{W}_{abcde} + A_{abcde})X^e = 0 \tag{16b}$$

$$\bar{X}_{abc[de]} + \hat{X}_{abc[de]} + B_{abc[de]f}\, X^f = 0 \tag{16c}$$

where

$$X = X^a \frac{\partial}{\partial x^a} + X_{abc} \frac{\partial}{\partial T_{abc}} + X_{abcd} \frac{\partial}{\partial Z_{abcd}} + \bar{X}_{abcde} \frac{\partial}{\partial \bar{W}_{abcde}} + \hat{X}_{abcde} \frac{\partial}{\partial \hat{W}_{abcde}}$$

is an unknown vector tangent to $\mathfrak{N}$ at Q. A direct inspection of the system (16) yields $s_0' = s_0$, while the vector $Y_{(1)}$, solution to (16), is given by

$$Y^{(1)}_{abc} = Z_{abcd}Y^d_{(1)} =: Z_{abc1}$$

$$Y^{(1)}_{abcd} = \bar{W}_{abcd1} + \hat{W}_{abcd1} + A_{abcd1}$$

$$\bar{Y}^{(1)}_{abcde} = \bar{Y}^{(1)n}_{abcde} + \bar{Y}^{(1)0}_{abcde} \tag{17a}$$

$$\hat{Y}^{(1)}_{abcde} = -\tfrac{2}{3}\,(B_{abe(cd)1} + \bar{Y}^{(1)n}_{abe(cd)} + \bar{Y}^{(1)0}_{abe(cd)}) + \sigma^{(1)}_{abcde} \tag{17b}$$

where the explicit expressions of the solutions (17a, b) to (16c) have been determined in the Appendix.

To proceed further, we have to enlarge the system (16) by adding the equations

$$\bar{X}_{abcde}Y^e_{(\alpha)} - \bar{Y}^{(\alpha)}_{abcde}X^e = 0 \tag{18a}$$

$$\hat{X}_{abcde}Y^e_{(\alpha)} - \hat{Y}^{(\alpha)}_{abcde}X^e = 0 \tag{18b}$$

where, for the present, $\alpha = 1$. Then the system (16), (18) satisfies the condition $s_1' = s_1$ and hence it is possible to determine a second vector $Y_{(2)}$, linearly independent of $Y_{(1)}$, as the general solution to (16), (18). Explicitly from (16) we find

$$Y^{(2)}_{abc} = Z_{abcd}Y^d_{(2)} =: Z_{abc2}$$

$$Y^{(2)}_{abcd} = \bar{W}_{abcd2} + \hat{W}_{abcd2} + A_{abcd2}$$

$$\bar{Y}^{(2)}_{abcde} = \bar{Y}^{(2)n}_{abcde} + \bar{Y}^{(2)0}_{abcde} \tag{19a}$$

$$\hat{Y}^{(2)}_{abcde} = -\tfrac{2}{3}\,(B_{abe(cd)2} + \bar{Y}^{(2)n}_{abe(cd)} + \bar{Y}^{(2)0}_{abe(cd)}) + \sigma^{(2)}_{abcde} \tag{19b}$$

while conditions (18) read

$$\bar{Y}^{(2)0}_{abcd1} = \bar{Y}^{(1)0}_{abcd2} + \bar{Y}^{(1)n}_{abcd2} - \bar{Y}^{(2)n}_{abcd1} \tag{20a}$$

$$\sigma^{(2)}_{abcd1} = \sigma^{(1)}_{abcd2} + \tfrac{2}{3}\,(B_{ab1(cd)2} - B_{ab2(cd)1} + \bar{Y}^{(2)n}_{ab1(cd)} - \bar{Y}^{(1)n}_{ab2(cd)} + \bar{Y}^{(2)0}_{ab1(cd)} - \bar{Y}^{(1)0}_{ab2(cd)}) \tag{20b}$$

Now the algebraic system to be considered consists of equations (16), (18), with $\alpha = 1, 2$. At this step, when determining s_2' and s_2, we arrive at a relation which must be identically true in order that $s_2' = s_2$. Specifically, multiplying (16c) by $Y^d_{(1)}Y^e_{(2)}$ and eliminating $\bar{X}_{abc[12]} - \hat{X}_{abc[21]}$ by means of (18) lead to

$$\bar{Y}^{(2)}_{abc1d} - \bar{Y}^{(1)}_{abc2d} + \hat{Y}^{(2)}_{abc1d} - \hat{Y}^{(1)}_{abc2d} + 2B_{abc[12]d} = 0 \tag{21}$$

As a rather lengthy but straightforward calculation shows, relation (21) results in an identity because of (17), (19), (20). This allows a vector $Y_{(3)}$, linearly independent of $Y_{(1)}$, $Y_{(2)}$, to be found as a solution to (16) and (18), with $\alpha = 1, 2$.

Analogously, the analysis of the system (16), (18), $\alpha = 1, 2, 3$, provides the last result, namely, $s_3' = s_3$. In conclusion the exterior system (14) is involutive with respect to $x^1, \ldots, x^4$, independently of the data N_{abcd} and of the geometry, thereby ensuring the existence of singular solutions to equation (2). ∎

It seems worthwhile to comment briefly on Lanczos' differential gauge which consists in the possibility of choosing the skew-symmetric tensor $T_{ab}{}^{e}{}_{;e}$ arbitrarily [1, 2]. Consistently with the statement (iii) of the theorem proved in I, it may be shown that the differential system formed by equation (2) and by an arbitrary condition on $T_{ab}{}^{e}{}_{;e}$ (e.g., $T_{ab}{}^{e}{}_{;e} = 0$) always admits singular solutions. This fact can easily be proved by a slight modification of the previous proof.

§(4): *Discussion*

In this second paper we have continued the investigation concerning the existence of third-order potentials for the Riemann and Weyl tensors. The first result, strengthening the theorem proved in I, is that equation (2) does never admit any ordinary solution (Theorem 1). Usually this does not prevent the existence of solutions at all, but rather suggests that solutions can be found provided the data satisfy suitable integrability conditions. The problem of determining them was left open in I. Here we have achieved the unexpected result that, in the four-dimensional case, no integrability condition is required (Theorem 2). In other words, looking at the class of singular solutions allows a third-order tensor potential to exist without any restriction on the given tensor N_{abcd} and on the geometric structure of the underlying Riemannian manifold.

We may now summarize the main result of this paper by saying that every tensor N_{abcd}, defined on an arbitrary four-dimensional Riemannian manifold and enjoying the symmetry properties (3), always admits a third-order tensor potential T_{abc} via equation (2). Of course the study of the representation (2) will be of special interest chiefly in connection with those tensor fields whose importance is already established on geometric (or physical) ground. Particular results have been obtained in conjunction with the Riemann tensor [7, 8] while the case of the Levi-Civita alternating tensor has been completely solved in I. Another nontrivial consequence of our analysis is that the Weyl tensor too allows the representation (2), besides the one discovered by Lanczos [2]; see also I, where it is shown that Lanczos' formula always admits ordinary solutions for the potential, and Ref. 9, where explicit Lanczos' representations have been found. As a further example, we observe that the tensor $g_{ac}g_{bd} - g_{ad}g_{bc}$, built up with the sole metric tensor g_{ab}, meets the symmetry properties (3) and then it has a potential representation; in this case multiplying equation (2) by g^{bd} yields the relation

$$3g_{ac} = \tfrac{1}{4}(T_a{}^b{}_{c;b} + T_c{}^b{}_{a;b} - T_a{}^b{}_{b;c} - T_c{}^b{}_{b;a})$$

whereby T_{abc} plays the role of a potential for the metric g_{ab} itself.

Another significant feature of the representation (2) is that, in a sense, it is valid irrespective of the choice of the affine connection. More precisely, simply by regarding the Γ's in (4a) and in (5) as the coefficients of an arbitrary affine connection ∇, Theorems 1 and 2 imply that the equation

$$N_{abcd} = \nabla_{\{d}T_{abc\}}$$

does always admit singular solutions, even though it never admits ordinary solutions.

Finally, we discuss briefly the existence of singular solutions to (2) in an n-dimensional Riemannian manifold (the case of ordinary solutions has been presented in I). Our conclusions depend strongly on the structure of equation (A6). Specifically, as follows from Theorem 2, when $n \leqslant 4$ there exist singular solutions to (2) independently of the choice of the tensor N_{abcd} and of the affine connection. When $n \geqslant 5$ equation (A6) places nontrivial restrictions on the data. In this instance, definite results can be obtained by studying the algebraic compatibility of (A6) and possibly by examining successive prolongations of the system (4).

Appendix

Consider the algebraic system

$$\bar{X}_{abc[de]} + \hat{X}_{abc[de]} + D_{abc[de]} = 0 \tag{A1}$$

where $D_{abcde} = D_{\{abcd\}e}$ is a given tensor, while

$$\bar{X}_{abcde} = \bar{X}_{[ab][cd]e} = -\bar{X}_{cdabe}, \qquad \hat{X}_{abcde} = \hat{X}_{[ab](cd)e}$$

are unknown tensors. Since $\hat{X}_{ab[cde]}$ vanishes identically, the system (A1) admits solutions provided $\bar{X}_{abcde}$ satisfies the condition

$$\bar{X}_{ab[cde]} + D_{ab[cde]} = 0 \tag{A2}$$

In view of (A2), the system (A1) may be cast into the alternative form

$$2\hat{X}_{abe[cd]} = \bar{X}_{abcde} + D_{abcde}$$

It is easy to verify that, on account of (A2), $\hat{X}_{abcde}$ may be expressed in terms of $\bar{X}_{abcde}$ and D_{abcde} through the relation

$$\hat{X}_{abcde} = -\tfrac{2}{3}\,(\bar{X}_{abe(cd)} + D_{abe(cd)}) + \sigma_{abcde} \tag{A3}$$

where $\sigma_{abcde} = \sigma_{[ab](cde)}$ is an arbitrary tensor. This shows that condition (A2) is also sufficient for determining the general expression of $\hat{X}_{abcde}$.

So as to discuss the system (A2), observe first that the relations

$$\bar{X}_{a[b|c|de]} - \bar{X}_{b[c|a|de]} = 0 \tag{A4}$$

$$D_{a[c|b|de]} + D_{b[c|a|de]} = 0 \tag{A5}$$

hold identically owing to the symmetry properties of $\bar{X}_{abcde}$ and D_{abcde}. A necessary condition on D_{abcde}, in order that the system (A2) be solvable, may be derived as follows. Perform on the indices $abcde$ of (A2) the operations $[abc]de$ and $a[b|c|de] - c[b|a|de]$. Then, after a suitable relabeling of the indices, compare the two expressions so obtained. The use of (A2), (A4), (A5) leads to the compatibility condition on the data D_{abcde}, namely,

$$D_{ab[cde]} + 2D_{[cde][ab]} + 2D_{b[c|a|de]} = 0 \tag{A6}$$

Whenever D_{abcde} satisfies (A6), the general solution to (A2) takes the form

$$\bar{X}_{abcde} = \bar{X}^{n}_{abcde} + \bar{X}^{0}_{abcde}$$

where

$$\bar{X}^{n}_{abcde} = \tfrac{2}{5}\,(D_{eacdb} - D_{ebcda} - D_{ecabd} + D_{edabc} + D_{e[ab][cd]} - D_{e[cd][ab]})$$

is a particular solution to (A2), while $\bar{X}^{0}_{abcde}$ is the general solution to $\bar{X}_{ab[cde]} = 0$.

To summarize, a necessary and sufficient condition for the existence of solutions to (A1) is that (A6) holds.

It is a striking feature of the condition (A6) that it reduces to an identity unless the indices $abcde$ are all different. As a consequence, when the dimension of the manifold is not greater than four, (A6) does not place any restriction on the data D_{abcde}.

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