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DISSIPATIVE MATERIALS AND VARIATIONAL FORMULATIONS

F. Bampi and A. Morro
Institute of Engineering Mathematics, Pl. Kennedy, 16129 Genova, Italy

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Introduction

A general theorem of Vainberg delivers necessary and sufficient conditions for an operator to admit a variational formulation [1] . A prominent application of Vainberg's theorem to systems of nonlinear differential equations leads to compatibility conditions (potentialness conditions) whose validity allows the differential equations at hand to be the Euler-Lagrange equations of a suitable functional [2-4] . It is commonly believed that the existence of a variational formulation, with a local-in-time Lagrangian, is peculiar to conservative systems. While it is usually so, there seems not to be a general argument ruling out the possibility of variational formulations for dissipative systems. It is the aim of this note to investigate this problem in connection with some materials which are currently viewed as dissipative bodies. Specifically, we look at materials of the Kelvin-Voigt-type for which the stress tensor is expressed through a nonlinear function of the deformation gradient, the velocity gradient, and also of the space and time variables. Then the potentialness conditions are applied so as to derive the most general form of constitutive functions admitting a variational formulation. Finally, on requiring that the stress function be objective and satisfy the standard symmetry conditions, we determine the Lagrangian density.

Potentialness Conditions

In order to describe finite motions we label each particle of the dissipative solid under consideration by its position $\mathbf{X}$ in a suitable reference configuration; so $\mathbf{x}(\mathbf{X},t)$ is the position of the particle $\mathbf{X}$ at time t, $\dot{\mathbf{x}} \equiv \mathbf{x}_{,t} = \partial \mathbf{x}(\mathbf{X},t)/\partial t$ is the velocity, $\mathbf{F} = \partial \mathbf{x}(\mathbf{X},t)/\partial \mathbf{X}$ is the deformation gradient.
Owing to the conservation of mass, the mass density ρ is determined through the

motion by $\rho \det \mathbf{F} = \rho_0(\mathbf{X})$, ρ_0 being the reference mass density. On accounting for the stress of the body through the first Piola-Kirchhoff stress tensor $\mathbf{S}$, the momentum balance reads

$$\rho_0 \ddot{\mathbf{x}} = \mathrm{Div}\,\mathbf{S} + \rho_0 \mathbf{b} \tag{1}$$

where $\mathbf{b} = \mathbf{b}(\mathbf{X},t)$ is the body force. As we are not interested in the temperature field we disregard the energy balance which, though, would lead to formidable difficulties in finding a genuine variational principle.

Our problem is to ascertain whether and how possible constitutive functions for the stress tensor $\mathbf{S}$ allow the balance equation (1) to arise from a variational principle. To accomplish this in a systematic way we need a proper mathematical tool. Now, on appealing to the general theory developed in [4] we state that a third-order system of the form

$$f_\Gamma(u_\Omega,\ u_{\Omega,\alpha},\ u_{\Omega,\alpha\beta},\ u_{\Omega,\alpha\beta\gamma},\ y_\alpha) = 0, \quad \Gamma,\Omega = 1,\dots,m, \quad \alpha,\beta,\gamma = 1,\dots,n, \tag{2}$$

in the unknown functions $u_\Omega = u_\Omega(y_\alpha)$, admits a variational formulation if and only if the potentialness conditions

$$\frac{\partial f_\Gamma}{\partial u_{\Omega,\alpha\beta\gamma}} = -\frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha\beta\gamma}}\,, \tag{3}$$

$$\frac{\partial f_\Gamma}{\partial u_{\Omega,\alpha\beta}} = \frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha\beta}} - 3\left(\frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha\beta\gamma}}\right)_{,\gamma}\,, \tag{4}$$

$$\frac{\partial f_\Gamma}{\partial u_{\Omega,\alpha}} = -\frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha}} + 2\left(\frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha\beta}}\right)_{,\beta} - 3\left(\frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha\beta\gamma}}\right)_{,\beta\gamma}\,, \tag{5}$$

$$\frac{\partial f_\Gamma}{\partial u_\Omega} = \frac{\partial f_\Omega}{\partial u_\Gamma} - \left(\frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha}}\right)_{,\alpha} + \left(\frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha\beta}}\right)_{,\alpha\beta} - \left(\frac{\partial f_\Omega}{\partial u_{\Gamma,\alpha\beta\gamma}}\right)_{,\alpha\beta\gamma} \tag{6}$$

are satisfied; the summation convention is in force and a comma followed by a greek letter, α say, denotes differentiation with respect to y_α, $\alpha = 1,\dots,n$. In this note the conditions (3)-(6) are applied in connection with the nonlinear Kelvin-Voigt model which, if the explicit dependence on $\mathbf{X}$ and t is allowed, is characterized by the response function

$$\mathbf{S} = \mathbf{S}(\mathbf{F}, \dot{\mathbf{F}}, \mathbf{X}, t)\ .$$

Restrictions on the Nonlinear Kelvin-Voigt Model

Henceforth we use Cartesian tensor notation, the convention that capital indices refer to material coordinates and lower case indices to present coordinates, and the convention that round (square) brackets, enclosing a pair of capital (or lower case) indices, denote symmetrization (skew-symmetrization). So as to apply (3)-(6) to the balance equation (1) we let $\Gamma,\Omega = 1,2,3$ and $\alpha,\beta,\gamma = 1,2,3,4$; then we set $u_i = x_i$ $(i = 1,2,3)$, $y_M = X_M$ $(M = 1,2,3)$, $y_4 = t$.

Before commencing the investigation of (1) we observe that some care is required in dealing with derivatives involving symmetric quantities; this problem occurs here because of the dependence on $\dot{\mathbf{F}}$. To avoid pitfalls we adopt the rule that derivatives are evaluated through the expression of differentials. For example, $dS_{iM} = (\partial S_{iM}/\partial x_{p,\alpha\beta})\, dx_{p,\alpha\beta} + \dots$ the dots denoting terms concerning other independent variables. If α may take the values 1,2,3 while $\beta = 4$ or viceversa we have $dS_{iM} = 2(\partial S_{iM}/\partial x_{p,Qt})dx_{p,Qt} + \dots$ because $x_{p,Qt} = x_{p,tQ}$. As a consequence we get $S_{iM,R} = 2(\partial S_{iM}/\partial x_{p,Qt})\, x_{p,QRt} + \dots$ whence

$$S_{iM,M} = 2\,\frac{\partial S_{i(M}}{\partial x_{p,Q)t}}\, x_{p,QMt} + \dots \quad .$$

Accordingly the balance equation (1) may be written as

$$E_i := \rho_0 x_{i,tt} - \frac{\partial S_{i(M}}{\partial x_{p,Q)}}\, x_{p,QM} - 2\,\frac{\partial S_{i(M}}{\partial x_{p,Q)t}}\, x_{p,QMt} - \frac{\partial S_{iM}}{\partial X_M} - \rho_0 b_i = 0 \,. \tag{7}$$

To ascertain whether (7) may admit a variational formulation we begin by deriving some restrictions placed by (3) and (4). Now, in view of (7),

$$\frac{\partial E_i}{\partial x_{p,QMt}} = -\,\frac{2}{3}\,\frac{\partial S_{i(M}}{\partial x_{p,Q)t}}\,;$$

this is so because $x_{p,QMt} = x_{p,QtM} = x_{p,tQM}$ and then

$$dE_i = \frac{\partial E_i}{\partial x_{p,\alpha\beta\gamma}}\, dx_{p,\alpha\beta\gamma} + \dots = 3\,\frac{\partial E_i}{\partial x_{p,QMt}}\, dx_{p,QMt} + \dots \,.$$

On the other hand we have

$$\frac{\partial E_i}{\partial x_{p,Qt}} = -\,\frac{\partial}{\partial x_{p,Qt}}\, S_{iM,M} = -\left(\frac{\partial S_{iM}}{\partial x_{p,Qt}}\right)_{,M}$$

and hence

$$\frac{\partial E_i}{\partial x_{p,Qt}} - 3\left(\frac{\partial E_i}{\partial x_{p,QMt}}\right)_{,M} = \left(\frac{\partial S_{iQ}}{\partial x_{p,Mt}}\right)_{,M}$$

Accordingly the conditions (3), (4) imply that

$$\frac{\partial S_{i(M}}{\partial x_{p,Q)t}} + \frac{\partial S_{p(M}}{\partial x_{i,Q)t}} = 0 \,, \tag{8}$$

$$\left(\frac{\partial S_{iM}}{\partial x_{p,Qt}} + \frac{\partial S_{pQ}}{\partial x_{i,Mt}}\right)_{,M} = 0 \,. \tag{9}$$

The integration of (9) yields at once

$$\frac{\partial S_{iM}}{\partial x_{p,Qt}} + \frac{\partial S_{pQ}}{\partial x_{i,Mt}} = K_{ipMQ} \tag{10}$$

where $K_{ipMQ} = K_{piQM}$ is an arbitrary function on $\mathbf{X}$,t and, obviously, $K_{ipMQ,M} = 0$. Comparison with (8) gives $K_{ip(MQ)} = 0$ and then $K_{ipMQ} = K_{[ip][MQ]}$. Therefore

$K_{[ip][MQ],M}=0$; hence there exists a tensor function $\Psi_{ip}=\Psi_{[ip]}$ such that

$$K_{ipMQ}=\varepsilon_{MQR}\,\Psi_{ip,R}$$

ε_{MQR} being the Levi-Civita symbol. Look now at (10); the homogeneous counterpart is a Killing equation ([5], §84), namely $\partial S_{iM}/\partial \dot{x}_{p,Q} + \partial S_{pQ}/\partial \dot{x}_{i,M} = 0$. Then the general solution to (10) is

$$S_{iM} = \sigma_{iM} + A_{iMpQ}\, x_{p,Qt} + \varepsilon_{MQR}\,\Psi_{ip,R}\, x_{p,Qt} \tag{11}$$

where $\sigma_{iM}=\sigma_{iM}(\mathbf{F},\mathbf{X},t)$ and $A_{iMpQ}=-A_{pQiM}$. In principle, A_{iMpQ} may depend on $\mathbf{F}$, besides on $\mathbf{X}$ and t; however, so as to arrive at definite results, we assume that A_{iMpQ} is independent of $\mathbf{F}$.

We have now to exploit the condition (5) and also (4) in connection with $\alpha,\beta=$ $=1,2,3$. Observe that

$$\frac{\partial E_i}{\partial x_{p,Q}} = -\frac{\partial}{\partial x_{p,Q}} S_{iM,M} = -\left(\frac{\partial S_{iM}}{\partial x_{p,Q}}\right)_{,M},$$

$$\frac{\partial E_i}{\partial x_{p,QM}} - 3\left(\frac{\partial E_i}{\partial x_{p,QMt}}\right)_{,t} = -\frac{\partial S_{i(M}}{\partial x_{p,Q)}} + 2\left(\frac{\partial S_{i(M}}{\partial x_{p,Q)t}}\right)_{,t},$$

$$-\frac{\partial E_i}{\partial x_{p,Q}} + 2\left(\frac{\partial E_i}{\partial x_{p,QM}}\right)_{,M} + 2\left(\frac{\partial E_i}{\partial x_{p,Qt}}\right)_{,t} - 6\left(\frac{\partial E_i}{\partial x_{p,QMt}}\right)_{,Mt} = \left[-\frac{\partial S_{iQ}}{\partial x_{p,M}} + 2\left(\frac{\partial S_{iQ}}{\partial x_{p,Mt}}\right)_{,t}\right]_{,M}.$$

Therefore the conditions (4), (5) yield

$$\frac{\partial S_{i(M}}{\partial x_{p,Q)}} - \frac{\partial S_{p(M}}{\partial x_{i,Q)}} + 2\left(\frac{\partial S_{p(M}}{\partial x_{i,Q)t}}\right)_{,t} = 0, \tag{12}$$

$$\frac{\partial S_{iM}}{\partial x_{p,Q}} - \frac{\partial S_{pQ}}{\partial x_{i,M}} + 2\left(\frac{\partial S_{pQ}}{\partial x_{i,Mt}}\right)_{,t} = H_{ipMQ} \tag{13}$$

where H_{ipMQ} is an arbitrary function on $\mathbf{X}$, t such that $H_{ipMQ,M}=0$. Comparison of (12) and (13) provides $H_{ipMQ} = H_{ip[MQ]}$. Substitution of (11) into (13) gives

$$\frac{\partial \sigma_{iM}}{\partial x_{p,Q}} - \frac{\partial \sigma_{pQ}}{\partial x_{i,M}} - A_{iMpQ,t} + \varepsilon_{MQR}\,\Psi_{ip,Rt} = H_{ipMQ}. \tag{14}$$

On interchanging the pairs iM, pQ and adding the two equations we obtain $H_{[ip][MQ]} = \varepsilon_{MQR}\,\Psi_{ip,Rt}$, whence $H_{[ip][MQ],M} = 0$. As a consequence we have

$$H_{(ip)[MQ],M} = 0. \tag{15}$$

Consequently there exists a symmetric tensor function $\Phi_{ip}=\Phi_{(ip)}$ on $\mathbf{X}$, t such that $H_{(ip)[MQ]} = \varepsilon_{MQR}\,\Phi_{ip,R}$. Upon substitution, (14) becomes

$$\frac{\partial \sigma_{iM}}{\partial x_{p,Q}} - \frac{\partial \sigma_{pQ}}{\partial x_{i,M}} = A_{iMpQ,t} + \varepsilon_{MQR}\,\Phi_{ip,R}.$$

Thus there exists a scalar function Σ on $\mathbf{F}$, $\mathbf{X}$, t such that

$$\sigma_{iM} = \frac{\partial \Sigma}{\partial x_{i,M}} + \frac{1}{2}A_{iMpQ,t}\,x_{p,Q} + \frac{1}{2}\varepsilon_{MQR}\,\Phi_{ip,R}\,x_{p,Q}.$$

Accordingly we arrive at the expression

$$S_{iM} = \frac{\partial \Sigma}{\partial x_{i,M}} + \frac{1}{2} A_{iMpQ,t}\, x_{p,Q} + \frac{1}{2} \varepsilon_{MQR}\, \Phi_{ip,R}\, x_{p,Q} + A_{iMpQ}\, x_{p,Qt} + \varepsilon_{MQR}\, \Psi_{ip,R}\, x_{p,Qt} \tag{16}$$

for the stress **S**. As the last step, upon a straightforward calculation we find that the function (16) satisfies identically the condition (6).

Observe that the above analysis does not involve the term $\rho_0 x_{i,tt}$ because it gives rise to identities only.

Objectivity and Symmetry Conditions

The balance equation (1), with the stress **S** as expressed by (16), admits a variational formulation. However, before determining the corresponding Lagrangian density, we observe that the expression (16) reduces severely if **S** is required to be objective [6] and the Cauchy stress to be symmetric, namely

$$S_{iM}\, x_{j,M} = S_{jM}\, x_{i,M} \, . \tag{17}$$

Of course, the scalar function Σ is objective provided that it depends on **F** through $\mathbf{C} = \mathbf{F}^T \mathbf{F}$. Accordingly, subsequent differentiations of (17) with respect to $x_{h,Nt}$ and $x_{k,R}$, and appeal to the symmetry properties of A_{iMpQ} and Ψ_{ip} lead to the results

$$A_{iMpQ} = 0 \, , \qquad \varepsilon_{MQR}\, \Psi_{ip,R} = 0 \, .$$

Next, subsequent differentiations of (17) with respect to $x_{h,N}$ and $x_{k,R}$ provide

$$\varepsilon_{MQR}\, \Phi_{ip,R}\, x_{p,Q} = 0 \, .$$

As a consequence, (16) reduces to

$$S_{iM} = \frac{\partial \Sigma}{\partial x_{i,M}} \, . \tag{18}$$

The Lagrangian Density

Now we move on to determine the Lagrangian density corresponding to (18). According to the general theory [4], systems of the form (2) admitting a variational formulation arise from the Lagrangian density

$$L(u_\Gamma) = (u_\Omega - \hat{u}_\Omega) \int_0^1 f_\Omega(\tilde{u}_\Gamma, \tilde{u}_{\Gamma,\alpha}, \tilde{u}_{\Gamma,\alpha\beta})\, d\xi \tag{19}$$

where $\hat{u}_\Omega$ is a fixed function and $\tilde{u}_\Omega = \hat{u}_\Omega + \xi(u_\Omega - \hat{u}_\Omega)$. Without any loss of generality we put $\hat{u}_i = x_i(\mathbf{X})$; then (19) becomes

$$L(\mathbf{x}) = \frac{1}{2}\rho_0 (x_i - \hat{x}_i) x_{i,tt} - \rho_0 (x_i - \hat{x}_i) b_i - \int_0^1 (x_i - \hat{x}_i) S_{iM,M}(\tilde{\mathbf{x}})\, d\xi \ ,$$

the dependence on $\mathbf{X}$, t being understood. Then, since

$$(x_i - \hat{x}_i)_{,M} \frac{\partial \Sigma}{\partial x_{i,M}} = \frac{\partial \Sigma}{\partial \xi} \ ,$$

an integration by parts yields, up to boundary terms (and up to a sign), the sought Lagrangian density

$$L(\mathbf{x}) = \frac{1}{2}\rho_0 \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} - \Sigma(\mathbf{F}) + \rho_0 \mathbf{x} \cdot \mathbf{b} \ . \tag{20}$$

The Lagrangian density (20) tells us that the function Σ plays the role of a potential energy. This feature is hardly surprising although usually, in continuum mechanics, the prescription $L = T - V$ only works for conservative particle -like systems. Indeed the structure of a particle-like system occurs just because the use of Lagrangian coordinates preserves the similarity with a system of discrete particles.

To sum up the results of this note we say that if the dissipative behaviour of a body is described through the Kelvin-Voigt model then the dependence on $\dot{\mathbf{F}}$ cannot be cast in a variational formulation. So the question arises about whether more involved models, like, for example, that of Zener ([7], §16), allows a variational formulation to account for the dissipation; this investigation is under way.

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