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Third-Order Tensor Potentials for the Riemann and Weyl Tensors

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Abstract

The representations of the Riemann and the Weyl tensors of a four-dimensional Riemannian manifold through covariant derivatives of third-order potentials are examined in detail. The Weyl tensor always admits a completely general representation whereas the Riemann tensor does not. Nevertheless there exists a class of Riemannian manifolds whose Riemann tensors may be calculated in terms of potentials; in this connection, specific examples are exhibited explicitly. The possibility of introducing gauges on the potentials is reexamined in connection with the previous result. New properties of the representations are also discussed.

§(1): *Introduction*

A more fundamental understanding of the geometric character of the gravitational theory has underlined the crucial role of the Riemann and the Weyl tensors. Gravitational radiation, singularity theory, geodesic deviation as a tool for experimental tests, to mention only a few, are topics which witness adequately the importance of the Riemann and the Weyl tensors in the realm of gravity. In turn, a deep knowledge of geometric aspects concerning the Riemann and the Weyl tensors provides a powerful and helpful insight into the modeling and the interpretation of the physical reality.

In accordance with the previous considerations, almost 20 years ago Lanczos [1] realized that the purely algebraic approach to the splitting of the Riemann tensor indicated by Einstein in [2] was unfortunately insufficient for "a true

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understanding of the structure of a four-dimensional Riemannian manifold." Meanwhile he looked at the problem from a novel viewpoint. The striking result of Lanczos' analysis was that the Weyl tensor C_{abcd} of any four-dimensional Riemannian manifold is generated, in a differential way, by a third-order tensor H_{abc} satisfying the symmetry properties

$$H_{abc} = H_{[ab]c}, \qquad H_{[abc]} = 0 \tag{1}$$

where, as usual, square brackets indicate antisymmetrization. Precisely, the differential link between C_{abcd} and H_{abc} is [1]

$$\begin{aligned} C_{abcd} = {} & H_{abc;d} - H_{abd;c} + H_{cda;b} - H_{cdb;a} - \tfrac{1}{2}[g_{ac}(H_{bd} + H_{db}) \\ & - g_{ad}(H_{bc} + H_{cb}) + g_{bd}(H_{ac} + H_{ca}) - g_{bc}(H_{ad} + H_{da})] \\ & + \tfrac{2}{3} H^{ef}{}_{e;f}(g_{ac}g_{bd} - g_{ad}g_{bc}), \end{aligned} \tag{2}$$

where $H_{bd} = H_b{}^e{}_{d;e} - H_b{}^e{}_{e;d}$, while g_{ab} denotes the metric tensor and a semicolon stands for the corresponding Riemannian covariant derivative. Actually Lanczos' result, completely equivalent to (2), follows from (2) by imposing the algebraic gauge $H_{ab}{}^b = 0$ [1] (cf. also Section 5).

The decomposition (2) was used by Avez [3] in order to characterize a special class of Einstein spaces while many years later, and precisely in 1975, Taub [4] developed the spinorial counterpart of Lanczos' work. Also, more recently, in an attempt to generalize Lanczos' viewpoint, Brinis Udeschini [5] argued that not only the Weyl tensor C_{abcd} but also the full Riemann tensor R_{abcd} could be derived from a potential H_{abc}, satisfying (1), through the formula

$$R_{abcd} = H_{abc;d} - H_{abd;c} + H_{cda;b} - H_{cdb;a} \tag{3}$$

Subsequently the same author elaborated the spinorial version of her result [6] and then drew the consequences of (3) in a number of distinguished applications [7, 8].

In our opinion, the significant results already achieved by means of the potential approach to the structure of the Riemann and Weyl tensors call for further investigation since we believe that the existence of a solution to either (2) or (3) is still to be proved. Precisely, what is proved in [1] is that equation (2) is the Euler–Lagrange equation of a suitable functional, while the deduction of (3), exhibited in [5], is based on the fact that there exists a functional of the field H_{abc} whose Euler–Lagrange equations take the form of the Bianchi identities for the tensor R_{abcd} built up according to (3). Nevertheless no rigorous proof that the equations so obtained are consistent is presented. On the other hand, there is no *a priori* reason for the consistency of the Euler–Lagrange equations of a given functional. A nontrivial example confirming this assertion may be found, e.g., in [9] where it is proved that a variational principle for gravity and matter in which metric and connection can be varied independently of each other leads, in general, to inconsistent field equations.

The aim of this paper is threefold. The primary aim is to decide unambiguously whether equations (2) and (3) admit a solution or not. To this end, in Section 2 we consider two problems, equivalent to (2) and (3), respectively, obtained by letting the Riemann and the Weyl tensors not obey the cyclic property. This allows us to avoid some algebraic troubles in the proof of the existence theorem presented in Section 3. The main result is that, whereas equation (2) always admits a solution, in general equation (3) does not. In addition, we are able to prove that the differential gauge on H_{abc} invoked by Lanczos in [1] can indeed be imposed not only in connection with equation (2), but also with respect to equation (3). Furthermore, the possible existence of solutions to (3) for very special choices of R_{abcd} is discussed in Section 4 where a few vacuum gravitational fields are exhibited for which solutions to (3) are explicitly calculated.

The second aim is to examine some general features of the representations (2), (3). Precisely, with the purpose of gaining new information, Section 5 is devoted to a detailed comparison between outstanding properties of (2) and the corresponding ones of (3). In so doing, besides deriving new results, we better realize the reciprocal role of (2) and (3). In passing we also investigate the existence problem for (2) and (3) in an n-dimensional Riemannian manifold, $n \neq 4$.

A third, minor, aim is to emphasize, through a prominent application, the effectiveness of Cartan's local criteria of integrability of ideals of exterior forms employed in Section 3.2 as a powerful tool for proving the main results of this paper. In this connection, it is worth pointing out that we use Cartan's local criteria in a noncovariant way; as a consequence, we are not able to derive explicitly the integrability conditions for equation (3), which, on the other hand, appear to be rather involved. Such conditions, however, are implicitly contained in the proof of the theorem presented here which also provides a necessary and sufficient procedure to ascertain the existence of a solution to (3) for any given R_{abcd}. A conjecture is then suggested that there exists a broad class of Riemann tensors admitting a third-order potential via (3). In any case, arriving at an exact solution to (2) or (3) is a very formidable task.

As a last comment, we observe that the paper is so organized that the proof of the theorem—which is in fact rather technical—may be omitted without loss of mathematical continuity.

§(2): *Two Equivalent Problems*

In discussing the existence of solutions to (2) or (3), some algebraic troubles arise in connection with the cyclic property of R_{abcd}, C_{abcd}, H_{abc}. Thus we are led to disregard the cyclic property by looking at two slightly different—but equivalent—problems.

Precisely, let N_{abcd} be a given tensor satisfying the symmetry relations

$$N_{abcd} = N_{[ab][cd]} = N_{cdab} \tag{4}$$

and define the trace-free part of N_{abcd} according to the formula

$$\begin{aligned} M_{abcd} &= N_{abcd} - \frac{1}{4}\{g_{ac}N_{bd}\} + \frac{N}{24}\{g_{ac}g_{bd}\} \\ &= N_{abcd} - \frac{1}{4}\left\{g_{ac}\left(N_{bd} - \frac{1}{6}Ng_{bd}\right)\right\} \end{aligned} \tag{5}$$

where $N_{bd} = g^{ac}N_{abcd}$ and $N = g^{bd}N_{bd}$. In (5) the braces are a short-hand notation for the following linear operation on the indices $(abcd)$:

$$\begin{aligned} \{(abcd)\} = (abcd) - (bacd) - (abdc) + (badc) \\ + (cdab) - (cdba) - (dcab) + (dcba) \end{aligned}$$

By analogy with (2), (3), we now introduce an unknown third-order tensor T_{abc}, satisfying the symmetry condition

$$T_{abc} = T_{[ab]c} \tag{6}$$

through either the equation

$$N_{abcd} = \{T_{abc;d}\} \tag{7}$$

or the equation

$$\begin{aligned} M_{abcd} = \{T_{abc;d}\} - \tfrac{1}{4}\{g_{ac}(T_b{}^e{}_{d;e} + T_d{}^e{}_{b;e} - T_b{}^e{}_{e;d} - T_d{}^e{}_{e;b})\} \\ + \tfrac{1}{6} T^{ef}{}_{e;f}\{g_{ac}g_{bd}\} \end{aligned} \tag{8}$$

It is important to emphasize that, although equation (8) can formally be derived from (7) via the definition (5), in the sequel we shall look at equations (7), (8) as being completely independent of each other.

The reason why we are allowed to examine equations (7), (8), instead of (2), (3), is contained in the following:

Lemma. The existence of a solution T_{abc} to equation (7) [(8)] is mathematically equivalent to the existence of a solution H_{abc} to equation (3) [(2)].

To prove this lemma, observe first that the tensors N_{abcd} and T_{abc} may uniquely be split as

$$N_{abcd} = \hat{N}_{abcd} + \lambda\eta_{abcd} \tag{9}$$

$$T_{abc} = \hat{T}_{abc} + \eta_{abcd}Q^d \tag{10}$$

where η_{abcd} is the Levi-Civita symbol and

$$\hat{N}_{abcd} = \tfrac{2}{3}\left(N_{abcd} + \tfrac{1}{2}N_{adcb} + \tfrac{1}{2}N_{acbd}\right), \qquad \hat{N}_{a[bcd]} = 0$$
$$\lambda = \tfrac{1}{24}\,\eta^{abcd}N_{abcd}$$
$$\hat{T}_{abc} = \tfrac{2}{3}\left(T_{abc} + \tfrac{1}{2}T_{cba} + \tfrac{1}{2}T_{acb}\right), \qquad \hat{T}_{[abc]} = 0$$
$$Q^d = \tfrac{1}{6}\,\eta^{abcd}T_{abc} \tag{11}$$

Similarly, we may write $M_{abcd} = \hat{M}_{abcd} + \mu\eta_{abcd}$, where $\hat{M}_{a[bcd]} = 0$. Then, in view of the linearity of (7)-(11), it is a simple matter to verify that $\hat{T}_{abc}$ and $\hat{N}_{abcd}$ satisfy equation (3) whenever T_{abc} and N_{abcd} satisfy equation (7) and that $\hat{T}_{abc}$ and $\hat{M}_{abcd}$ satisfy equation (2) whenever T_{abc} and M_{abcd} satisfy equation (8). Moreover, on account of (9), (10), antisymmetrization of (7) with respect to bcd yields

$$\lambda\eta_{abcd} = \{\eta_{abce}Q^e{}_{;d}\} \tag{12}$$

whereby

$$\lambda = Q^e{}_{;e} \tag{13}$$

The same equation is arrived at also by starting from (8). Since equation (13) always admits a (local) solution, we conclude that the existence of solutions to (3) [(2)] is a consequence of the existence of solutions to (7) [(8)]. The converse of this statement is trivially true.

As a preliminary check on the equations we are considering, let us observe that, owing to the symmetry properties of the tensors involved, the number of equations against the number of unknowns is as follows:

system (7):	21 equations, 24 unknowns
system (3):	20 equations, 20 unknowns
system (8):	11 equations, 24 unknowns
system (2):	10 equations, 20 unknowns

For a better understanding of the previous table, it should be borne in mind that the skew-symmetric part (13) of (7) or (8) is a single equation for four independent quantities. Moreover, it seems that a solution to equation (2) or (8) enjoys a great arbitrariness which could be eliminated by imposing suitable gauges. The algebraic and differential gauges introduced in the literature [1, 5] will be examined in the next sections.

§(3): *The Existence Theorem*

3.1. Statement of the Theorem. In the sequel we shall refer to generic conditions as the assumption that the field N_{abcd} (and possibly M_{abcd}) is constrained only to meet the algebraic symmetry properties (4) and no further *a priori* restriction. Also, in the subsequent analysis we shall suppose that the underlying manifold and the given data N_{abcd}, or M_{abcd}, are analytic; this hypothesis enables us to investigate the existence of a solution to (7) or (8) through the use of Cartan's local criteria of integrability of ideals of exterior forms (formal approaches to Cartan's local criteria may be found, e.g., in [10, 11], while a more intuitive treatment is delivered in [12]). Accordingly our conclusions have a local character only.

The main results to emerge from this section are collected in the following.

Theorem. Consider an analytic four-dimensional Riemannian manifold and let N_{abcd} and M_{abcd} be analytic tensor fields satisfying the symmetry properties (4) and $M^e{}_{bec} = 0$. Then (i) under generic conditions there does not exist any local regular analytic solution to equation (7); (ii) for every choice of the field M_{abcd} equation (8) always admits a local regular analytic solution; (iii) the system consisting of equation (8) and the differential gauge condition

$$T_{ab}{}^{e}{}_{;e} = 0 \tag{14}$$

always admits a local regular analytic solution. Analogously, whenever one local regular analytic solution to equation (7) exists, then the system (7), (14) too admits a local regular analytic solution.

3.2. Proof of the Theorem. (i) The proof is based on a preliminary reformulation of the system of partial differential equations (7) in terms of exterior differential forms and on the use of Cartan's necessary and sufficient condition for involutiveness [10]. To this end, let us define the auxiliary quantities

$$Z_{abcd} = Z_{[ab]cd} := T_{abc,\,d} \tag{15}$$

where a comma denotes partial differentiation with respect to the local coordinates x^d of the Riemannian manifold. The system (7) may thus be written explicitly in the equivalent form

$$N_{abcd} - \{Z_{abcd} - \Gamma^e{}_{ad} T_{ebc} - \Gamma^e{}_{bd} T_{aec}\} = 0 \tag{16}$$

while the definition of Z_{abcd} implies that

$$dT_{abc} - Z_{abce}\, dx^e = 0 \tag{17}$$

Consider now the array (x^a, T_{abc}, Z_{abcd}) as the local coordinates of a formal 124-dimensional (analytic) manifold $\mathfrak{M}$. Accordingly, we shall look at equations (16), (17) as a system of exterior differential equations on $\mathfrak{M}$. In agreement with this viewpoint, the addition of the equations obtained by exterior differentiation of (16), (17), viz.,

$$N_{abcd,\,f}\, dx^f + \{(\Gamma^e{}_{ad,\,f} T_{ebc} + \Gamma^e{}_{bd,\,f} T_{aec})\, dx^f - dZ_{abcd} + \Gamma^e{}_{ad}\, dT_{ebc} + \Gamma^e{}_{bd}\, dT_{aec}\} = 0 \tag{18}$$

$$dZ_{abce} \wedge dx^e = 0 \tag{19}$$

leads to the system (16)-(19) which is closed under exterior differentiation and has the same integral manifolds in $\mathfrak{M}$ as the original system (16), (17) [10-12]. Obviously, every solution to the original system (7) singles out in $\mathfrak{M}$ a four-dimensional integral manifold of the system (16)-(19) which is, by definition, involutive with respect to $x^1, \ldots, x^4$. It is the central point of Cartan's approach to partial differential equations that every four-dimensional integral manifold of (16)-(19), involutive with respect to $x^1, \ldots, x^4$—and hence given locally in $\mathfrak{M}$ by $T_{abc} = T_{abc}(x^1, \ldots, x^4)$, $Z_{abcd} = Z_{abcd}(x^1, \ldots, x^4)$—identifies itself with a solution to (7).

Our task is then to look for the existence of four-dimensional integral manifolds of (16)-(19) which are involutive with respect to $x^1, \ldots, x^4$. According to Cartan's theorem (see [10], p. 91), the necessary and sufficient condition for an integral manifold to be involutive is that the reduced Cartan characters s_i' coincide with the corresponding Cartan characters s_i, that is, $s_i' = s_i$. In essence, the characters s_i and s_i' are ranks of algebraic linear systems; such characters are to be evaluated through a step-by-step procedure which, in the present case, goes as follows.

Choose a point $P = (x^a, T_{abc}, Z_{abcd})$ of $\mathfrak{M}$ in such a way that equation (16) holds at P. Letting $X = X^a(\partial/\partial x^a) + X_{abc}(\partial/\partial T_{abc}) + X_{abcd}(\partial/\partial Z_{abcd})$ be an unknown vector belonging to the tangent space to $\mathfrak{M}$ at P, we consider both the algebraic linear system [generated by the 1-forms (17), (18)]

$$X_{abc} - Z_{abce}X^e = 0 \tag{20}$$

$$N_{abcd,f}X^f + \{(\Gamma^e{}_{ad,f}T_{ebc} + \Gamma^e{}_{bd,f}T_{aec})X^f - X_{abcd} + \Gamma^e{}_{ad}X_{ebc} + \Gamma^e{}_{bd}X_{aec}\} = 0 \tag{21}$$

and the reduced linear system obtained from (20), (21) simply by omitting the terms containing the quantities X^a. Then s_0 and s_0' are defined to be the rank of (20), (21) and of the reduced system corresponding to (20), (21), respectively. Consider now the further equation [generated by the 2-form (19)]

$$X^d Y_{(\alpha)abcd} - Y^d{}_{(\alpha)} X_{abcd} = 0 \tag{22}$$

where, for the present, $\alpha = 1$ and $Y_{(1)}$ is a known vector solution to (20), (21). Then $s_0 + s_1$ and $s_0' + s_1'$ are the rank of (20)-(22) and of the reduced system corresponding to (20)-(22), respectively. Let $Y_{(2)}$ be a solution to the system (20)-(22) independent of $Y_{(1)}$. Then $s_0 + s_1 + s_2$ and $s_0' + s_1' + s_2'$ are the rank of (20)-(22), $\alpha = 1, 2$, and of the reduced system corresponding to (20)-(22), $\alpha = 1, 2$, respectively. As a last step, we have to find the vector $Y_{(3)}$ as a solution of (20)-(22), $\alpha = 1, 2$, independent of $Y_{(1)}$, $Y_{(2)}$; hence the construction proceeds straightaway. It is, however, worthwhile to remind the reader that at each step the vector $Y_{(\alpha)}$ must be so chosen as to depend on the maximum number of independent parameters consistent with the system yielding $Y_{(\alpha)}$: no further constraint on these parameters may be imposed in subsequent steps.

Let us draw the consequences of the procedure indicated above by looking explicitly at the evaluation of s_2 and s_2'. At this step we have at our disposal two independent vectors $Y_{(1)}$ and $Y_{(2)}$; by multiplying the reduced equation corresponding to (21) by $Y^a{}_{(1)}\,Y^b{}_{(2)}\,Y^c{}_{(1)}\,Y^d{}_{(2)}$ we readily recognize that, on account of (20), (22), equation (21) reduces to an identity. On the other hand, when applied to the full system, the previous calculation leads to

$$\begin{aligned}[N_{abcd,f}X^f + \{(\Gamma^e{}_{ad,f}T_{ebc} + \Gamma^e{}_{bd,f}T_{aec} - \Gamma^e{}_{ad}Z_{ebcf} - \Gamma^e{}_{bd}Z_{aecf})\,X^f\}]\\ \cdot Y^a{}_{(1)}\,Y^b{}_{(2)}\,Y^c{}_{(1)}\,Y^d{}_{(2)} + 4(Y_{(1)abcf}Y^c{}_{(2)}\\ - Y_{(2)abcf}Y^c{}_{(1)})\,Y^a{}_{(1)}\,Y^b{}_{(2)}\,X^f = 0\end{aligned} \tag{23}$$

which is a homogeneous linear equation for X^a. Under generic conditions equation (23) is not identically satisfied; as a consequence we have $s_2' < s_2$ or, more precisely, $s_2 = s_2' + 1$. The general result is therefore that, whereas under generic conditions the full system does not contain any internal identity, the reduced system contains as many internal identities as the number of independent fourth-order tensors meeting the symmetry properties (4) which can be formed by means of the three independent vectors $Y_{(1)}$, $Y_{(2)}$, $Y_{(3)}$. Since this number is six, we deduce that $s_0 + s_1 + s_2 + s_3 = s_0' + s_1' + s_2' + s_3' + 6$.

This discussion enables us to conclude that, under generic conditions, the system (16)-(19) does not admit any involutive regular integral manifold and hence equation (7) too does not admit any local regular analytic solution.

(ii) The proof follows exactly along the same lines as that of part (i); consequently we only summarize the main steps. Introduce first the auxiliary variables Z_{abcd} via equation (15), then rewrite equation (8) as a system of exterior differential equations on $\mathfrak{M}$ and make it closed under exterior differentation. So we are in a position to calculate the Cartan characters s_i and s_i'. As all steps are easily reproducible, it is sufficient to point out that the presence of the terms $\{g_{ac}(T_b{}^e{}_{d;e} + T_d{}^e{}_{b;e}\}$ in (8) now prevents the formation of the internal identities plaguing the reduced system in part (i). Hence, in the present case, $s_i = s_i'$. Accordingly we conclude that equation (8) always admits a local regular analytic solution.

(iii) With the aid of the definition (15), carry the first equation (14) over $\mathfrak{M}$ to get

$$g^{cd}(Z_{abcd} - \Gamma^e{}_{ad}T_{ebc} - \Gamma^e{}_{bd}T_{aec} - \Gamma^e{}_{cd}T_{abe}) = 0 \tag{24}$$

Next calculate its exterior derivative, namely,

$$\begin{aligned}g^{cd}(dZ_{abcd} - \Gamma^e{}_{ad}\,dT_{ebc} - \Gamma^e{}_{bd}\,dT_{aec} - \Gamma^e{}_{cd}\,dT_{abe}) + g^{cd}{}_{,f}[(Z_{abcd}\\ - \Gamma^e{}_{ad}T_{ebc} - \Gamma^e{}_{bd}T_{aec} - \Gamma^e{}_{cd}T_{abe}) - g^{cd}(\Gamma^e{}_{ad,f}T_{ebc}\\ + \Gamma^e{}_{bd,f}T_{aec} + \Gamma^e{}_{cd,f}T_{abe})]\,dx^f = 0\end{aligned} \tag{25}$$

Finally consider the enlarged system consisting of equations (8)—or rather (8) and its closure in $\mathfrak{M}$—(24), and (25).

Now we apply Cartan's theorem. Everything goes as in part (ii) provided the enlarged system is algebraically consistent. It is easy to convince oneself that this is so. For, multiplication of (8) by g^{cd} results in an identity thereby showing that equations (24), (25) do not participate in the operation contained in (8). This feature stands in complete agreement with the heuristic reason for the validity of the gauge (14) given by Lanczos [1]. The conclusion is then that the enlarged system satisfies $s_i = s_i'$, thus ensuring the existence of involutive integral manifolds and, in turn, the existence of local regular analytic solutions to (8), (14).

An analogous result holds in connection with the system (7), (14) provided the conditions for an involutive integral manifold to exist are satisfied, that is, when the data N_{abcd} are not generic.

§(4): *Discussion*

According to the preceding theorem, under generic conditions equation (7) does not admit any solution; of course such a theorem does not prevent the existence of (nongeneric) tensors N_{abcd} which can be expressed in terms of third-order tensor potentials T_{abc} via equation (7). Yet, when this is true, equation (7) looks rather formidable to be solved. In this section we shall discuss a few particular examples which allow a solution of (7) to be found explicitly.

To begin with, look at the flat space-time manifold and define the tensor N_{abcd} through the relation

$$N_{abcd} = \tfrac{1}{2}\left(\gamma_{ad,\,bc} + \gamma_{bc,\,ad} - \gamma_{ac,\,bd} - \gamma_{bd,\,ac}\right) \tag{26}$$

where γ_{ab} is an arbitrary second-order symmetric tensor. It is worth observing that (26) is the expression of the Riemann tensor in the linearized theory. In this instance it is known that a solution to (7) is [1, 7]

$$T_{abc} = \tfrac{1}{4}\left(\gamma_{bc,\,a} - \gamma_{ac,\,b}\right) \tag{27}$$

This is a simple example of a tensor which is not the Riemann tensor of the underlying (flat) manifold and which nevertheless admits a tensor potential (further details on this example are given in [7]).

Consider now the case of a conformally flat manifold. Owing to the vanishing of the Weyl tensor, the Riemann tensor turns out to be expressed as [cf. equation (5)]

$$R_{abcd} = \tfrac{1}{4}\,\{g_{ac}(R_{bd} - \tfrac{1}{6}\,Rg_{bd})\} \tag{28}$$

thereby showing that the tensor $R_{bd} - \frac{1}{6} R g_{bd}$ determines completely R_{abcd} (see, e.g., [13]). Then, on letting the metric tensor be given by

$$g_{ab} = \exp\,[2U(x^1, \ldots, x^4)]\ \eta_{ab}, \quad \eta_{ab} = \mathrm{diag}(1, 1, 1, -1)$$

we eventually arrive at [13]

$$R_{ab} - \tfrac{1}{6} R g_{ab} = -2U_{,ab} + 2U_{,a} U_{,b} - \eta_{ab} U_{,e} U^{,e} \tag{29}$$

We now seek solutions to (3) of the form

$$H_{abc} = f_{,a} g_{bc} - f_{,b} g_{ac} \tag{30}$$

$f = f(x^1, \ldots, x^4)$; as follows straightaway by substitution of (30) into (2), such a solution does not alter the vanishing of the Weyl tensor. According to (28)-(30), equation (3) reduces to

$$(-U + 2f)_{,ab} + (U - 4f)_{,a} U_{,b} + (U - 4f)_{,a} U_{,b} - \eta_{ab}(U - 4f)_{,e} U^{,e} = 0 \tag{31}$$

A solution to the previous system is obtained by letting

$$f = \tfrac{1}{4} U$$

hence (31) becomes

$$U_{,ab} = 0$$

whence

$$U = \lambda_a x^a$$

λ_a being a vector constant with respect to η_{ab}.

Furthermore, we present two exact conformally flat solutions of the Einstein field equations whose Riemann tensors may be derived from potentials.

First, look at the most general conformally flat solution with pure radiation or electromagnetic null fields. In this instance the energy momentum tensor is

$$T_{ab} = \Phi^2(u)\, u_{,a} u_{,b}$$

where $u_{,a} u^{,a} = 0$, $u_{;ab} = 0$, and the line element reads [13]

$$ds^2 = dx^2 + dy^2 - 2\, du\, dv - \tfrac{1}{2} \Phi^2(u)(x^2 + y^2)\, du^2$$

Seeking solutions of the form (30), with $f = f(u)$, reduces (3) to

$$-4f_{,u} u_{,a} u_{,b} = \Phi^2 u_{,a} u_{,b}$$

thus the potential is given by (30), $f(u)$ being a solution to

$$4f_{,u} = \Phi^2$$

Second, we examine conformally flat perfect fluid solutions with state equation $p = -\mu/3$. The relevant energy momentum tensor is

$$T_{ab} = \frac{2}{3}\mu t_{,a}t_{,b} - \frac{\mu}{3}g_{ab}$$

where $\mu = \mu(t)$, $t_{,a}t^{,a} < 0$, and $t_{;ab} = \frac{1}{3}\theta(t)(t_{,a}t_{,b} + g_{ab})$, while the line element turns out to be a special case of the generalized Friedmann universes (see, e.g., [13], p. 370). As above, we find that solutions of the form (30) are allowed provided the function $f(t)$ satisfies the equation

$$3f_{,t} + f\theta = 0$$

As a final remark, we note in passing that the Ricci tensor η_{abcd} of every Riemannian manifold can always be derived from a potential. Indeed equation (12) shows that η_{abcd} is given in terms of the potential $\eta_{abcd}Q^d$, where Q^d is a solution to $Q^d{}_{;d} = 1$.

§(5): *Comments and Conclusions*

We are now in a position to draw a detailed comparison between the two differential problems represented by (7) and (8). In particular, we shall indicate some properties which either have already been proved or are easily verified by direct computation.

Consider first equation (8); we have the following results.

(a_1) For every assigned tensor M_{abcd} equation (8) does always admit a solution.

(a_2) Every solution to (8) is determined to within the tensor $V_a g_{bc} - V_b g_{ac}$, V_a being an arbitrary vector. This algebraic gauge has been used to make $T_{ab}{}^b$ vanish [1, 5].

(a_3) For every solution to (8) the skew-symmetric tensor $T_{ab}{}^e{}_{;e}$ may be fixed arbitrarily. This constitutes a differential gauge which in general cannot be expressed in a local form. However, it should be noted that, in the case of a manifold with constant curvature, every skew-symmetric tensor ψ_{ab} yields the quantity $\psi_{ab;c}$ which makes the right-hand side of (8) vanish identically and which may be viewed as the local counterpart of the aforementioned differential gauge.

As to equation (7), the corresponding results are as follows.

(b_1) Equation (7) may allow a solution for particular choices of the tensor N_{abcd}; when this is the case, a solution to (8) may be found through the relation (5) [5].

(b_2) Whenever the Riemannian manifold admits a Killing vector k_a, a possible solution to (7) is determined to within the tensor $k_a g_{bc} - k_b g_{ac}$.

(b_3) The same as (a_3). As a further comment, we strongly believe that it is just the presence of such a general differential gauge–involving six degrees of freedom–which is responsible for the six internal identities arising in the proof of the theorem and leading to the generic nonexistence of solutions to (7).

We conclude this paper with two remarks. First, by the very nature of the problem at hand, the vanishing of N_{bd} neither provides any significant simplification nor makes the six internal identities of the theorem hold automatically. In the framework of the general relativity this means that there may exist vacuum solutions of the Einstein field equations whose Riemann tensors cannot be given the form (3). Needless to say that, when equation (3) has a solution, the conclusions arrived at in [7, 8] yield relevant and outstanding insights into the geometric structure of the Einstein field equations.

Second, it seems worthwhile to analyze the problem of the existence of a solution to (7) or (8) in an n-dimensional Riemannian manifold, with n not necessarily equal to four. As to equation (7) is follows directly from the proof of the theorem that only for $n = 2$ equation (7) always admits a solution whereas for $n \geqslant 3$ at least one internal identity appears thereby preventing the existence of a solution under generic conditions. Things are different for equation (8). In fact there arise algebraic incompatibilities only when the number of independent equations becomes greater than the number of independent unknowns. Now, in view of their symmetry properties, M_{abcd} has $\alpha_1 = (n^4 - 2n^3 - n^2 - 6n)/8$ and T_{abc} has $\alpha_2 = (n^3 - n^2)/2$ algebraically independent components. It is readily recognized that $\alpha_1 < \alpha_2$ when $n \leqslant 6$ and that $\alpha_1 > \alpha_2$ when $n \geqslant 7$. Accordingly, we conclude that, under generic conditions, equation (8) admits a solution if and only if $n \leqslant 6$.

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