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OBJECTIVE CONSTITUTIVE RELATIONS IN ELASTICITY AND VISCOELASTICITY

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SOMMARIO. Si esamina la compatibilità tra il principio di obiettività ed equazioni costitutive affini per i tensori di stress elastici di Cauchy a Piola-Kirchhoff con stress residuo non nullo. Generalizzando risultati di Fosdick e Serrin si prova che il tensore di Cauchy può essere soltanto un tensore costante, proporzionale al tensore identità, mentre il tensore di Piola-Kirchhoff può essere una funzione lineare del gradiente di deformazione. Alle stesse conclusioni si perviene anche partendo dal funzionale della viscoelasticità. Infine si mostra che, per materiali tipo Maxwell, le soluzioni di equazioni di evoluzione obiettive sono funzionali obiettivi.

SUMMARY. The compatibility between the objectivity principle and affine constitutive equations for the elastic Cauchy and Piola-Kirchhoff stress tensors with non-zero residual stress is examined. It is found that the Cauchy stress is allowed to be only a constant tensor, proportional to the identity tensor, while the Piola-Kirchhoff stress may be a linear function on the deformation gradient thus generalizing previous results by Fosdick and Serrin. The same conclusions are arrived at also by starting from viscoelasticity. Finally, in the case of Maxwell-like materials, the solutions to the objective evolution equations are shown to be objective functionals.

1. INTRODUCTION.

For a constitutive theory to be a mathematical model representing a material behaviour in an acceptable way, a suitable set of general requirements must be satisfied [1, 2]. Among them, the axiom of objectivity (or material frame-indifference) has been given much attention also because of many objections against objectivity as a principle. Moreover, attention has been focused on the axiom of objectivity so as to derive restrictions on the constitutive relations. In this regard, enlightening comments and valuable results may be found in [3 - 5].

Lately, Fosdick and Serrin [6] have investigated the consequences of objectivity on the constitutive relations for elastic solids. Indeed, they have considered the question of linear elasticity with zero residual stress as being a theory or nothing but an useful approximation. As a result, compatibility with objectivity has been shown to demand that the response function should vanish; this provides a noticeable feature which weights in favour of linear elasticity as being simply a useful approximation.

The purpose of this paper is twofold. The first fold is to re-examine compatibility of linear elasticity with objectivity when the residual stress is allowed to be non-zero. To this end, section 2 summarizes some properties concerning objective functions and exhibits assumptions about Cauchy stress and Piola-Kirchhoff stress as affine functions on the displacement gradient. Then, in section 3, restrictions placed by objectivity on the response functions are derived. Specifically, it turns out that the Cauchy stress must be proportional to the identity tensor while the Piola-Kirchhoff stress must be a linear function on the deformation gradient. So the objective relation between the Piola-Kirchhoff stress and the deformation gradient involves a symmetric constant tensor and then anisotropies of the body are described through six elastic constants, at the most. Moreover it is shown that the same conclusions are obtained even though one starts from affine viscoelasticity. All these results generalize the corresponding ones obtained by Foskick and Serrin in the case of zero residual stress.

Motivated by the view of viscoelasticity as a model arising from a differential equation accounting for relaxation (Maxwell's material), the second fold is to give new insights into the topic of objective functionals. In this regard, in section 4 we consider the most general objective form of a Maxwell-like differential equation and we show, by a direct procedure, that the functional so determined is objective. To our mind the connection between objective functionals and evolution equations involving objective time derivatives, proved here in a particular case, could be an interesting subject for future investigations.

2. AFFINE CONSTITUTIVE RELATIONS.

Henceforth we describe the deformation of the body under consideration through the deformation gradient tensor $\mathbf{F}$, or the displacement gradient tensor $\mathbf{H}$, relative to some fixed reference configuration of the body; letting $\mathbf{1}$ be the identity tensor, $\mathbf{F}$ and $\mathbf{H}$ are related by $\mathbf{F} = \mathbf{H} + \mathbf{1}$. Both $\mathbf{H}$ and $\mathbf{F}$ are elements of the space L of all linear transformations of a three dimensional vector space into itself. So as to avoid the vanishing of the local volume elements or mirror reflections of the reference configuration, we require that $\mathbf{H} \in D := \{\mathbf{H} \in L,\ \det(\mathbf{1} + \mathbf{H}) > 0\}$. Indeed,

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it is enough to assume that $\mathbf{H} \in D'$, D' being any open connected subset of D which contains the origin $\mathbf{0}$. Physically this corresponds to look only at displacement gradients which can be achieved by actual deformations from the undistorted state. At any particle $\mathbf{X}$ and time t the elastic response of the body is expressed via the Cauchy stress $\mathbf{T}$ as

$$\mathbf{T} = \hat{\mathbf{T}}[\mathbf{H}]$$

or via the (first) Piola-Kirchhoff stress tensor $\mathbf{S} = |\det \mathbf{F}|\ \mathbf{T}\mathbf{F}^{-1T}$ as

$$\mathbf{S} = \hat{\mathbf{S}}[\mathbf{H}]$$

the functions $\hat{\mathbf{T}}$ and $\hat{\mathbf{S}}$ being defined on D'. Denoting the set of all proper orthogonal tensors by $\mathcal{O}^+$, for any time dependent tensor $\mathbf{Q} : \mathbf{R} \to \mathcal{O}^+$ the change of frame [1]

$$\mathbf{x} \to \mathbf{Q}\mathbf{x} + \mathbf{c} \tag{2.1}$$

causes the change of the deformation gradient $\mathbf{F} \to \mathbf{Q}\mathbf{F}$ and then, under (2.1), the displacement gradient transforms as $\mathbf{H} \to \mathbf{Q} - \mathbf{1} + \mathbf{Q}\mathbf{H}$. Thus, according to the axiom of objectivity, the functions $\hat{\mathbf{T}}, \hat{\mathbf{S}}$ must satisfy the conditions

$$\hat{\mathbf{T}}[\mathbf{Q} - \mathbf{1} + \mathbf{Q}\mathbf{H}] = \mathbf{Q}\hat{\mathbf{T}}[\mathbf{H}]\mathbf{Q}^T, \tag{2.2}$$

$$\hat{\mathbf{S}}[\mathbf{Q} - \mathbf{1} + \mathbf{Q}\mathbf{H}] = \mathbf{Q}\hat{\mathbf{S}}[\mathbf{H}], \tag{2.3}$$

for any tensor $\mathbf{H} \in D'$ and any (time dependent) tensor $\mathbf{Q} \in \mathcal{O}^+$ such that $\mathbf{Q} - \mathbf{1} + \mathbf{Q}\mathbf{H} \in D'$.

In this paper we confine our attention to affine functions $\hat{\mathbf{T}}$, $\hat{\mathbf{S}}$, namely

$$\hat{\mathbf{T}}[\mathbf{H}] = \mathbf{A} + \mathbf{B}[\mathbf{H}], \tag{2.4}$$

$$\hat{\mathbf{S}}[\mathbf{H}] = \mathbf{M} + \mathbf{N}[\mathbf{H}], \tag{2.5}$$

where $\mathbf{A}, \mathbf{M}$ are constant tensors and $\mathbf{B}, \mathbf{N}$ are linear functions on D'; of course $\mathbf{B}[\mathbf{0}] = \mathbf{0}$, $\mathbf{N}[\mathbf{0}] = \mathbf{0}$. In next section we exploit the conditions (2.2) - (2.5) so as to derive restrictions placed by objectivity. To this end, it is convenient to have recourse to the following

THEOREM 1 (Fosdick and Serrin [6]). *If the linear tensor function* $\mathbf{f} \in \mathrm{Hom}\ (L, L)$ *is such that* $\mathbf{f}[\mathbf{Q} - \mathbf{1}] = \mathbf{0}$ *for any tensor* $\mathbf{Q} \in \mathcal{O}^+$, $\mathbf{Q} - \mathbf{1} \in D'$, *then* $\mathbf{f}$ *vanishes identically on* L.

It is a trivial matter to write the counterparts of (2.2) - - (2.5) in the case of response functionals. For instance, letting

$$\mathcal{H} := \{\mathbf{H}(\cdot) : \mathbf{R} \to L,\ \det\,[\mathbf{1} + \mathbf{H}(\cdot)] > 0\},$$

the counterpart of (2.5) is

$$\mathbf{S}(t) = \mathcal{S}^{\,t}_{-\infty}[\mathbf{H}(\cdot)] = \mathbf{M} + \mathcal{N}^{\,t}_{-\infty}[\mathbf{H}(\cdot)],\ \mathbf{H}(\cdot) \in \mathcal{H}, \mathcal{N}^{\,t}_{-\infty}[\mathbf{0}(\cdot)] = \mathbf{0}. \tag{2.6}$$

In connecion with functionals on $\mathcal{H}$ we have the following

THEOREM 2 (Fosdick and Serrin [6]). *If the locally linear tensor functional* f *on* $\mathcal{H}$ *is such that* $f^{t}_{-\infty}[\mathbf{Q}(\cdot) - \mathbf{1}(\cdot)] = \mathbf{0}$ *for any tensor function* $\mathbf{Q}(\cdot) : \mathbf{R} \to \mathcal{O}^+$, $\mathbf{Q} - \mathbf{1} \in \mathcal{H}$, *then* f *vanishes identically on* $\mathcal{H}$.

3. LINEAR ELASTICITY AND VISCOELASTICITY WITH NON-ZERO RESIDUAL STRESS.

3.1. Objective Cauchy Stress Tensor.

The consequences of objectivity on the Cauchy stress tensor function $\hat{\mathbf{T}}$ are now achieved by means of a procedure which generalizes the one used by Truesdell [5]. To begin with, express the conditions (2.2), (2.4) in terms of $\mathbf{F}$; it follows that the relation

$$\mathbf{Q}(\mathbf{A} + \mathbf{B}[\mathbf{F}] - \mathbf{B}[\mathbf{1}])\mathbf{Q}^T = \mathbf{A} + \mathbf{B}[\mathbf{Q}\mathbf{F}] - \mathbf{B}[\mathbf{1}] \tag{3.1}$$

must be true for all invertible tensors $\mathbf{F}$, such that $\mathbf{F} - \mathbf{1} \in D'$, and all proper orthogonal tensors $\mathbf{Q}$ such that $\mathbf{Q}\mathbf{F} - \mathbf{1} \in D'$. Hence, choosing $\mathbf{F} = \gamma\mathbf{1}$, $\gamma \neq 0$, implies that

$$\mathbf{Q}\mathbf{A}\mathbf{Q}^T - \mathbf{A} = \gamma\mathbf{B}[\mathbf{Q}] - \mathbf{B}[\mathbf{1}] + (1 - \gamma)\mathbf{Q}\mathbf{B}[\mathbf{1}]\mathbf{Q}^T \tag{3.2}$$

must hold for any γ such that $(\gamma - 1)\mathbf{1} \in D'$. Now, it is an easy matter to recognize that eq. (3.2) holds if and only if

$$\mathbf{B}[\mathbf{Q}] = \mathbf{Q}\mathbf{B}[\mathbf{1}]\mathbf{Q}^T.$$

Then, in view of Appendix A, letting $\mathbf{E} = \mathbf{1}$ we obtain $\mathbf{B}[\mathbf{Q}] = \mathbf{0}$. Accordingly, eq. (3.2) becomes

$$\mathbf{Q}\mathbf{A}\mathbf{Q}^T - \mathbf{A} = \mathbf{0}$$

which shows that $\mathbf{A}$ commutes with any orthogonal matrix and thereafter implies that $\mathbf{A} = \alpha\mathbf{1}$, $\alpha \in \mathbf{R}$. On account of these results, eq. (3.1) reduces to

$$\mathbf{Q}\mathbf{B}[\mathbf{F}]\mathbf{Q}^T = \mathbf{B}[\mathbf{Q}\mathbf{F}].$$

Hence, on appealing again to Appendix A and letting $\mathbf{E} = \mathbf{F}$, we arrive at $\mathbf{B}[\mathbf{F}] = \mathbf{0}$. We can thus assert that the most general tensor function $\hat{\mathbf{T}}$ meeting the conditions (2.2), (2.4) is

$$\mathbf{T}[\mathbf{H}] = \alpha\mathbf{1}, \qquad \alpha \in \mathbf{R}. \tag{3.3}$$

This amounts to saying that a non-vanishing, objective, affine Cauchy stress tensor consists of an isotropic residual stress only. Although this requirement allows us to account for a non-vanishing Cauchy stress tensor, thereby generalizing Fosdick and Serrin's result, it seems that it is too restrictive to let linear elasticity be a theory rather than a useful approximation. It goes without saying that the restrictive result (3.3) hinges on the validity of the objectivity principle.

3.2. Objective Piola-Kirchhoff stress tensor.

Look at the conditions (2.3), (2.5) and set $\mathbf{H} = \mathbf{0}$; it follows at once that

$$\mathbf{N}[\mathbf{Q} - \mathbf{1}] = (\mathbf{Q} - \mathbf{1})\mathbf{M}. \tag{3.4}$$

No generality is lost by letting the linear function $\mathbf{N}$ be given the form

$$\mathbf{N}[\mathbf{H}] = \mathbf{H}\mathbf{M} + \mathbf{f}[\mathbf{H}],$$

where, for consistency with (3.4), the linear function $\mathbf{f}$ is such that $\mathbf{f}[\mathbf{Q} - \mathbf{1}] = \mathbf{0}$. We can then apply Fosdick and Serrin's theorem 1 to obtain that $\mathbf{f}[\mathbf{H}]$ must vanish identically. So $\mathbf{N}[\mathbf{H}] = \mathbf{H}\mathbf{M}$ and therefore the Piola-Kirchhoff stress tensor turns out to be objective if and only if

$$\mathbf{S}[\mathbf{H}] = (\mathbf{H} + \mathbf{1})\mathbf{M} = \mathbf{F}\mathbf{M}. \tag{3.5}$$

An immediate comparison between equations (3.3) and (3.5) shows that, while $\mathbf{T}$ can be at most a constant isotropic stress tensor, the existence of a residual stress allows $\mathbf{S}$ to be a non-trivial linear function on $\mathbf{F}$. In either case, however, as we should expect the result of Fosdick and Serrin is recovered by letting $\mathbf{A} = 0$ or $\mathbf{M} = 0$.

On account of (3.5), $\mathbf{M}$ coincides with the second Piola-Kirchhoff stress tensor and then it must be symmetric. So, in general, eq. (3.5) determines $\mathbf{S}$ through six elastic moduli. Of course the effective number of elastic constants may reduce because of the symmetry of the material under consideration. In particular, if the material is isotropic the Cauchy stress function $\hat{\mathbf{T}}$ must meet the condition [4]

$$\hat{\mathbf{T}}[\mathbf{F}] = \hat{\mathbf{T}}[\mathbf{FQ}], \qquad \forall \mathbf{Q} \in \mathcal{O}. \tag{3.6}$$

Now, since $\mathbf{T} = |\det \mathbf{F}|^{-1}\mathbf{FMF}^T$, the condition (3.6) implies that

$$\mathbf{M} = m\mathbf{I}, \qquad m \in \mathbf{R}. \tag{3.7}$$

In passing, we mention that the objective ansatz (3.7), with m as a function of a suitable hidden variable, has been considered by Kosinski [7] in connection with shock wave propagation in rheological continua.

3.3. Objective viscoelasticity.

In order that the functional (2.6) for the Piola-Kirchhoff stress meets objectivity requirements [1] the condition

$$\mathbf{Q}(t)\mathbf{M} + \mathbf{Q}(t)\mathcal{N}^{\,t}_{-\infty}\,[\mathbf{H}(\cdot)] = \mathbf{M} + \mathcal{N}^{\,t}_{-\infty}[\mathbf{Q}(\cdot) - \mathbf{I}(\cdot) + (\mathbf{QH})(\cdot)]$$

where $\mathbf{H}(\cdot) \in \mathcal{H}$ and $\mathbf{Q}(\cdot): \mathbf{R} \to \mathcal{O}^+$, must hold. Hence, on letting $\mathbf{H}(\cdot) = \mathbf{0}(\cdot)$ we have

$$\mathcal{N}^{\,t}_{-\infty}[\mathbf{Q}(\cdot) - \mathbf{I}(\cdot)] = (\mathbf{Q}(t) - \mathbf{I})\mathbf{M}. \tag{3.8}$$

Introduce now the functional f on $\mathcal{H}$ described by

$$\mathcal{N}^{\,t}_{-\infty}[\mathbf{H}(\cdot)] = \mathbf{H}(t)\mathbf{M} + f^{\,t}_{-\infty}[\mathbf{H}(\cdot)];$$

of course, in view of (3.8), f is subject to

$$f^{\,t}_{-\infty}[\mathbf{Q}(\cdot) - \mathbf{I}(\cdot)] = 0.$$

So, on appealing to theorem 2 we conclude that f vanishes identically and then

$$\mathbf{S}(t) = \mathbf{F}(t)\mathbf{M}. \tag{3.9}$$

As a result, starting from the functional (2.6) for viscoelastic bodies we have shown that, owing to objectivity, the Piola-Kirchhoff stress tensor must in fact be a linear function on $\mathbf{F}$. Indeed, the function (3.9) coincides with the function (3.5) which has been obtained in the context of elasticity. Accordingly, viscoelasticity is an outstanding example for showing how severe the restrictions placed by objectivity are.

It is an easy matter to see that, in the case of the viscoelastic Cauchy stress

$$\mathbf{T}(t) = \mathbf{A} + \mathcal{B}^{\,t}_{-\infty}[\mathbf{H}(\cdot)], \qquad \mathbf{H}(\cdot) \in \mathcal{H},$$

a strictly analogous procedure leads to the trivial constitutive equation $\mathbf{T} = \alpha\mathbf{I}$.

4. OBJECTIVE VISCOELASTICITY FROM OBJECTIVE EVOLUTION EQUATIONS.

Owing to the cumbersome calculations usually associated with models based on memory functionals, often the viscoelastic behaviour of materials is described through stress-strain relations involving flows as well. In the simplest case such relations have the form

$$\tau\dot{\mathbf{\Pi}} + \mathbf{\Pi} = \mathbf{P}, \qquad \mathbf{\Pi}(t_0) = \mathbf{\Pi}^0, \tag{4.1}$$

where the superposed dot denotes the material time derivative and $\tau > 0$ plays the role of relaxation time. For instance, eq. (4.1) accounts for Maxwell's materials by letting $\mathbf{\Pi}$ be the stress and $\mathbf{P}$ the material time derivative of the strain and for Kelvin-Voigt solids by letting $\mathbf{\Pi}$ be the strain and $\mathbf{P}$ the stress [8]. Besides, equations like (4.1) are also encountered as evolution equations for materials with hidden variables [9]; in such a case $\mathbf{\Pi}$ is the hidden variable and $\mathbf{P}$ the observable variable governing the growth of $\mathbf{\Pi}$ [10, 11].

Even though $\mathbf{\Pi}$ and $\mathbf{P}$ transform as objective tensors under (2.1), the presence of the material time derivative makes eq. (4.1) be a non-objective relation. This inconvenience may be overcome by replacing the dot derivative with an objective time derivative — henceforth denoted by a superposed spot. To this end we recall that the most general form of an objective time derivative acting on an objective mixed tensor $A^{i..}{}_{j..}$ is [12]

$$\begin{aligned}\mathring{A}^{i..}{}_{j..} &= \dot{A}^{i..}{}_{j..} + \Xi^{i}{}_{p}A^{p..}{}_{j..} + \ldots - \\ &- \Xi^{p}{}_{j}A^{i..}{}_{p..} - \ldots\end{aligned} \tag{4.2}$$

where $\mathbf{\Xi} = \mathbf{\Sigma} - \mathbf{\Lambda}$, $\mathbf{\Sigma}$ being an arbitrary objective tensor and $\mathbf{\Lambda} = -\mathbf{\Lambda}^T$ a non-objective tensor whose transformation law under (2.1) is

$$\mathbf{\Lambda} \to \mathbf{Q\Lambda Q}^T + \dot{\mathbf{Q}}\mathbf{Q}^T. \tag{4.3}$$

Two customary examples of objective time derivatives are the co-rotational and the convected ones [13]; they correspond to setting $\mathbf{\Sigma} = 0$, $\mathbf{\Lambda} = -\mathbf{W}$ and $\mathbf{\Sigma} = -\mathbf{D}$, $\mathbf{\Lambda} = -\mathbf{W}$, respectively, $\mathbf{D}$ being the stretching tensor and $\mathbf{W}$ the spin tensor. It is worth remarking that, in view of (4.2), an objective time derivative need not commute with raising and lowering of tensor indices; for instance, the co-rotational derivative commutes whereas the convected one does not.

The objective version of (4.1) is

$$\tau\mathring{\mathbf{\Pi}} + \mathbf{\Pi} = \mathbf{P}, \qquad \mathbf{\Pi}(t_0) = \mathbf{\Pi}^0. \tag{4.4}$$

For the sake of definiteness, we assume that $\mathbf{\Pi}$ and $\mathbf{P}$ are second rank objective controvariant tensors; then eq. (4.4) explicitly reads

$$\dot{\mathbf{\Pi}} - \mathbf{\Lambda\Pi} + \mathbf{\Pi\Lambda} + \mathbf{\Sigma\Pi} + \mathbf{\Pi\Sigma}^T + \frac{1}{\tau}\mathbf{\Pi} = \frac{1}{\tau}\mathbf{P}, \; \mathbf{\Pi}(t_0) = \mathbf{\Pi}^0. \tag{4.5}$$

Equation (4.5) is an objective differential relationship among $\mathbf{\Pi}$, $\mathbf{P}$, $\mathbf{\Sigma}$, and $\mathbf{\Lambda}$; its solution may be viewed as a functional

$$\mathbf{\Pi}(t) = \mathcal{P}^{\,t}_{-\infty}[\mathbf{P}(\cdot), \mathbf{\Sigma}(\cdot), \mathbf{\Lambda}(\cdot)] \tag{4.6}$$

on the functions $\mathbf{P}(\cdot)$, $\mathbf{\Sigma}(\cdot)$, $\mathbf{\Lambda}(\cdot)$. It is our goal to investigate the objective properties of the solution (4.6) to (4.5) as a functional. To this end we derive first an appropriate expression for the restrictions placed on $\mathcal{P}$ by the objectivity principle, according to which

$$Q(t)\mathscr{P}_{t_0}^{t}[P(\cdot), \Sigma(\cdot), \Lambda(\cdot)]Q^T(t) =$$

$$= \mathscr{P}_{t_0}^{t}[(QPQ^T)(\cdot), (Q\Sigma Q^T)(\cdot), (Q\Lambda Q^T + \dot{Q}Q^T)(\cdot)] \qquad (4.7)$$

must be true for any proper orthogonal tensor function Q on $[t_0, t]$. Define a second rank tensor function **Y** as solution of

$$\dot{Y} = \Lambda Y, \quad Y(t_0) = 1; \qquad (4.8)$$

as shown in the Appendix B, **Y** is an orthogonal tensor functional on $\Lambda(\cdot)$. Then setting $Q = Y^T$ in (4.7) provides

$$\mathscr{P}_{t_0}^{t}[P(\cdot), \Sigma(\cdot), \Lambda(\cdot)] =$$

$$= Y(t)\mathscr{P}_{t_0}^{t}[(Y^T P Y)(\cdot), (Y^T \Sigma Y)(\cdot), 0(\cdot)]\, Y^T(t) \qquad (4.9)$$

which makes it evident that $\mathscr{P}$ depends on the non-objective tensor Λ only through **Y**. On the other hand, if $\mathscr{P}$ meets eq. (4.9), the objectivity of **P** and Σ and eq. (B2) imply that $\mathscr{P}$ in objective. Hence, eq. (4.9) is a necessary and sufficient condition for $\mathscr{P}$ to be objective.

The last step of our investigation is now to determine whether eq. (4.6) agrees with eq. (4.9). To this purpose we must integrate eq. (4.5) so as to get the explicit expression for the functional $\mathscr{P}$. On defining the non-singular tensor J through the relations

$$\dot{J} = (Y^T \Sigma Y)J, \quad J(t_0) = 1, \qquad (4.10)$$

a straightforward but lengthy calculation yields the solution to (4.5) as

$$\Pi(t) = Y(t)(J^T)^{-1}(t)\Big\{\Pi(t_0)\exp[-(t-t_0)/\tau] +$$

$$+ \tau^{-1}\int_{t_0}^{t} \exp[(\xi - t)/\tau]\, J^T(\xi)Y^T(\xi)P(\xi)Y(\xi)J(\xi)\,d\xi\Big\} J^{-1}(t)Y^T(t). \qquad (4.11)$$

Accordingly, since J is a functional on $(Y^T\Sigma Y)$ $(\cdot)$ (cfr. Appendix B), it turns out that the functional (4.11) is a particular case of (4.9) and therefore, as we should expect, the solution to the objective differential equation (4.5) is an objective functional.

APPENDIX A

Look at the linear function B mapping D' into L such that

$$B_{ij}(E) = b_{ijhk}E_{hk}, \quad i, j = 1, 2, 3. \qquad (A\ 1)$$

Suppose further that, for a fixed E, B satisfies the condition

$$B[QE] = QB[E]\,Q^T \qquad (A\ 2)$$

for every $Q \in \mathcal{O}^+$, $QE \in D'$. As we show below, (A 2) implies $B[E] = 0$. In view of (A 1), condition (A 2) reads

$$b_{ijhk}Q_{hp}E_{pk} = Q_{ip}Q_{jq}b_{pqhk}E_{hk}. \qquad (A\ 3)$$

As D' is open, derivation of (A 3) with respect to Q_{rs} is allowed; it turns out that

$$b_{ijrk}E_{sk} = \delta_{ir}b_{sqhk}Q_{jq}E_{hk} + \delta_{jr}b_{pshk}Q_{ip}E_{hk}.$$

Hence, multiplication by Q_{rs} yields

$$b_{ijrk}Q_{rs}E_{sk} = 2Q_{ip}Q_{jq}b_{pqhk}E_{hk},$$

namely

$$B[QE] = 2QB[E]Q^T.$$

The immediate comparison with (A 2) leads to

$$B[QE] = 0, \quad Q \in \mathcal{O}^+,$$

and then

$$B[E] = 0.$$

APPENDIX B

Letting $K = K(t)$ be an arbitrary time dependent second. rank tensor, look at the first order differential equation

$$\dot{Z} = KZ. \qquad (B\ 1)$$

Equation (B 1) has a unique solution such that $Z(t_0)$ assumes an assigned value Z_0 [5]; such a solution is given by a functional

$$Z(t) = \mathscr{Z}_{t_0}^{t}[K(\cdot), Z_0]$$

on K. In view of the identity

$$(\det Z)^{\cdot} = (\det Z)\,\mathrm{tr}(\dot{Z}Z^{-1}),$$

account of (B 1) implies that

$$(\det Z)^{\cdot} = (\det Z)\,\mathrm{tr}\,K.$$

Hence $\det Z(t) \neq 0$ provided that $\det Z(t_0) \neq 0$ and then non-singular initial value tensors lead to non-singular solutions to eq. (B 1).

Consider now the particular case $K = -K^T$. Letting $\mathbb{Z} = ZZ^T$, in view of (B 1) we find that

$$\dot{\mathbb{Z}} = K\mathbb{Z} - \mathbb{Z}K.$$

On appealing to the uniqueness theorem for ordinary differential equations the condition $\mathbb{Z}(t_0) = 1$ implies that $\mathbb{Z}(t) = 1$ for all t. Accordingly, a solution to (B 1) which is orthogonal at $t = t_0$ is orthogonal for all t.

Let $\dot{Z}^* = K^*Z^*$ be the counterpart of eq. (B 1) under the transformation (2.1). Introduce now the further assumption that $K^* = QKQ^T + \dot{Q}Q^T$ (cfr. eq.(4.5)). On the other hand, on account of (B 1) we find that $K^* = (QZ)^*(QZ)^T$, that is to say

$$(QZ)^* = K^*(QZ), \quad K^* = -K^{*T}.$$

Hence, on appealing again to the uniqueness theorem, we assert that

$$Z^* = QZ \qquad (B\ 2)$$

provided $Z^*(t_0) = Q(t_0)Z(t_0)$ holds.

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