Mixture Approach to Plasmas with Dissipative Phenomena.

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(ricevuto il 21 Maggio 1981)

Summary. — On appealing to the recent well-established theory of mixtures, plasmas are described in a systematic way as mixtures of interacting charged-fluid constituents. Viscosity and heat conduction of the constituents are accounted for through the formalism of hidden variables so as to allow wave front propagation. A thermodynamic analysis is performed by having recourse to the second law in the form of the Clausius-Duhem inequality, thus providing a thorough scheme of dissipative effects in plasmas. For example, as an outstanding result of the investigation, it follows that, in the case of constituents at different temperatures, a transfer of linear momentum is unavoidably associated with a correspondent transfer of energy. It is then shown how, in the limiting approximation of magnetofluidodynamics, the process of electric conduction may be embodied into the framework of dissipative effects.

1. – Introduction.

The very important role played by plasmas is due to the fact that the major part of the Universe is in the plasma state. While this motivates the wide literature on the subject (1), the large amount of experimental observations at our disposal makes us try to set up models realistic enough to account for

(1) See, *e.g.*, B. LEHNERT: *Suppl. Nuovo Cimento*, **13**, 59 (1959); P. A. STURROCK, Editor: *Plasma Astrophysics, Proceedings S.I.F.*, Course XXXIX (New York, N. Y., 1967); N. A. KRALL and A. W. TRIVELPIECE: *Principles of Plasma Physics* (New York, N. Y., 1973); H. DEMIRAY and A. C. ERINGEN: *Plasma Phys.*, **15**, 903 (1973); R. Z. SEGDEEV: *Rev. Mod. Phys.*, **51**, 1, 11 (1979); G. SCHMIDT: *Physics of High Temperature Plasmas* (New York, N. Y., 1979), and references therein.

the plasma behaviour on cosmical and laboratory scale ([2]). Among the amazing applications of plasma models we mention that the features of high-frequency electromagnetic waves propagating in a magnetized plasma may reveal the existence of a black hole in a binary system ([3]).

Differently from what happens in magnetofluidodynamics, in a plasma, particularly at low densities, the standard description in macroscopic terms becomes inadequate and then we have to account for contributions arising from the individual particle behaviour and to regard the plasma as two or more coupled fluids coexisting in the same space. In other words, the plasma is to be viewed as a mixture of fluids. Of course, this outlook is not at all new in the literature ([4-6]). Yet, to our mind, within the context of macroscopic approaches, the present state of the theory of plasmas as mixtures is far from being conclusive. Besides this, we recall that in recent years the theory of mixtures has been definitely settled as a systematic chapter of continuum physics—see, *e.g.*, ref. ([7]). On the basis of these observations new improvements of plasma theory are likely to be gained by studying the plasma behaviour from the mixture standpoint. Indeed, new results are expected to arise from a detailed analysis of the energy balance and of the second law of thermodynamics.

It is a significant step along the road to a realistic model of plasmas to introduce the phenomena of heat conduction and viscosity ([8]) for the constituents of the mixture. To do this we cannot rely upon the Navier-Stokes-Fourier theory because, as is well known, it rules out the possibility of wave front propagation. Among the various descriptions of viscosity and heat conduction which remedy this shortcoming, the hidden-variable approach is recommended by its own flexibility ([9,10]). Beyond this satisfactory feature, two reasons at least justify the recourse to the hidden-variable tool. First, continuum theories are applicable to a good approximation at low densities provided

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([2]) H. ALFVÉN: *Phys. Today*, **24**, 28 (1971).

([3]) R. A. BREUER and J. EHLERS: *Proc. R. Soc. London Ser. A*, **370**, 389 (1980).

([4]) L. SPITZER: *Physics of Fully Ionized Gases* (New York, N. Y., 1962).

([5]) A. N. KAUFMAN: *Dissipative effects*, in *Plasma Physics in Theory and Application*, edited by W. B. KUNKEL (New York, N. Y., 1966).

([6]) P. C. CLEMMOW and J. P. DOUGHERTY: *Electrodynamics of Particles and Plasmas* (Reading, Mass., 1969).

([7]) R. M. BOWEN: *Theory of mixtures*, in *Continuum Physics*, edited by A. C. ERINGEN, Vol. III (New York, N. Y., 1976).

([8]) The viscosity of a fully ionized gas was analysed by S. I. BRAGINSKII: *Sov. Phys. JETP*, **6**, 358 (1958), who showed that viscosity is due primarily to positive ions, the viscous stresses due to electrons being generally negligible.

([9]) See, *e.g.*, F. BAMPI and A. MORRO: *J. Math. Phys. (N. Y.)*, **21**, 1201 (1980).

([10]) B. D. COLEMAN and M. E. GURTIN: *J. Chem. Phys.*, **47**, 597 (1967); W. A. DAY: *Arch. Ration. Mech. Anal.*, **62**, 367 (1976); A. MORRO: *Arch. Mech.*, **32**, 145 (1980).

that suitable relaxation terms are taken into account ([11]). Second, hidden variables have been proved to be a relevant formalism for describing at a macroscopic level typical behaviours of microscopic theories ([12]).

The plan of the paper is as follows. In sect. **2** we review briefly the general balance equations of a reacting mixture and then we apply them to plasmas. To account for viscosity and heat conduction, in sect. **3** the plasma is given the structure of material with hidden variables. The constitutive theory is developed by starting from assumptions general enough and by examining their compatibility with the second law of thermodynamics in the form of the Clausius-Duhem inequality. As a particular case, sect. **4** delivers a prominent constitutive theory which is the most direct extension to plasmas of a nonstationary Navier-Stokes-Fourier–like theory ([13]).

Two significant applications are presented in the second part of the paper. First, we are accustomed to considering energy transfers originated from differences of temperatures ([5]); sect. **5** shows that, whenever the constituents of the mixture are at different temperatures, a transfer of linear momentum is unavoidably associated with a correspondent transfer of energy arising from differences of kinetic energies. This result, reasonable on physical grounds, follows as a straightforward consequence of the theory of mixtures. Second, sect. **6** exhibits an approach to magnetofluidodynamics as a suitable limit of plasma physics. Such an approach gives new insights into the method of accounting for dissipative effects in nonstationary magnetofluidodynamics ([14]).

2. – Balance equations.

Roughly speaking a plasma is a multicomponent fluid whose constituents are electrons, neutral atoms and one or more types of ions. Accordingly, we idealize a plasma as a reacting mixture M of $\nu + 2$ fluid components labelled by the suffix $\alpha = -1$ (electrons), 0 (neutral atoms), $1, \ldots, \nu$ (ions). Thus, if we let n^α be the number density of the α-th constituent and $-e$ stand for the electron charge, the quantity $\alpha n^\alpha e$ represents the charge density. The particles of the α-th constituent are identified with their position $\boldsymbol{X}^\alpha$ in a suitable reference configuration. As usual, at any time t each place $\boldsymbol{x}$ in M is assumed to be occupied simultaneously by particles of all $\nu + 2$ constituents. A backward prime affixed to a symbol with a suffix α denotes the material time derivative following the motion of the α-th constituent. For example, $\boldsymbol{v}^\alpha = \boldsymbol{x}^{`\alpha} = \partial \boldsymbol{x}^\alpha(\boldsymbol{X}^\alpha, t)/\partial t$ is the velocity. The symbol ∇ stands for the spatial gradient

([11]) M. Carrassi and A. Morro: *Nuovo Cimento B*, **9**, 321 (1972); **13**, 281 (1973).
([12]) A. Morro: *Int. J. Eng. Sci.*, **18**, 913 (1980).
([13]) A. Morro: *Rend. Sem. Mat. Univ. Padova*, **64**, 59 (1980).
([14]) F. Bampi and A. Morro: *J. Non-Equilib. Thermodyn.*, **6**, 1 (1981).

operator, while $\boldsymbol{P}:\boldsymbol{Q}$ denotes the complete contraction of the second-order tensors $\boldsymbol{P}$, $\boldsymbol{Q}$, *viz.* $\boldsymbol{P}:\boldsymbol{Q} = \operatorname{tr}(\boldsymbol{P}\boldsymbol{Q}^{\mathrm{T}})$, $\boldsymbol{Q}^{\mathrm{T}}$ being the transpose of $\boldsymbol{Q}$.

Letting m^α be the weight of the particles of the α-th constituent, the balance equation for mass can be written as

$$m^\alpha(n^{\cdot\alpha} + n^\alpha \nabla\cdot\boldsymbol{v}^\alpha) = \hat{c}^\alpha\,, \tag{2.1}$$

where $\hat{c}^\alpha$ is the mass supply. Multiplication of (2.1) by $\alpha e/m^\alpha$ gives the balance equation for charge, namely

$$\alpha e(n^{\cdot\alpha} + n^\alpha \nabla\cdot\boldsymbol{v}^\alpha) = \frac{\alpha e\hat{c}^\alpha}{m^\alpha}\,. \tag{2.2}$$

Of course, conservation of mass and charge for the mixture as a whole implies that

$$\sum_\alpha \hat{c}^\alpha = 0\,, \qquad \sum_\alpha \alpha\hat{c}^\alpha/m^\alpha = 0\,. \tag{2.3}$$

Before stating the balance equations for linear momentum, angular momentum and energy in the case of plasmas, look briefly at the corresponding general relations for a mixture ([15]). The specific surface force is determined through the stress tensor $\boldsymbol{T}^\alpha$, while the body force accounts for the external contribution $n^\alpha m^\alpha \boldsymbol{f}^\alpha$ and for the momentum supply $\hat{\boldsymbol{p}}^\alpha$; thus the balance of linear momentum may be written as

$$n^\alpha m^\alpha \boldsymbol{v}^{\cdot\alpha} = \nabla\cdot\boldsymbol{T}^\alpha + \hat{\boldsymbol{p}}^\alpha + n^\alpha m^\alpha \boldsymbol{f}^\alpha\,, \tag{2.4}$$

where the divergence operator acts on the second index. Naturally, the supplies $\hat{\boldsymbol{p}}^\alpha$'s are subject to the condition

$$\sum_\alpha (\hat{\boldsymbol{p}}^\alpha + \hat{c}^\alpha \boldsymbol{u}^\alpha) = 0\,, \tag{2.5}$$

the diffusion velocity $\boldsymbol{u}^\alpha$ being defined by

$$\boldsymbol{u}^\alpha = \boldsymbol{v}^\alpha - \Big(\sum_\alpha n^\alpha m^\alpha \boldsymbol{v}^\alpha\Big)\Big/\Big(\sum_\alpha n^\alpha m^\alpha\Big)\,.$$

The balance of angular momentum results in the symmetry of the inner stress

([15]) A different look at balance equations for mixtures is delivered in appendix A. For a comprehensive account of the theory of mixtures the interested reader is referred to ref. ([7]).

tensor $\boldsymbol{T} = \sum_\alpha \boldsymbol{T}^\alpha$. More specifically, if $\boldsymbol{T}^\alpha_+$ and $\boldsymbol{T}^\alpha_-$ stand for the symmetric and the skew-symmetric parts of $\boldsymbol{T}^\alpha$, we have

$$\sum_\alpha \boldsymbol{T}^\alpha_- = 0 . \tag{2.6}$$

If we denote by $n^\alpha m^\alpha r^\alpha$, $\boldsymbol{q}^\alpha$ and $\hat{\varepsilon}^\alpha$ the external heat supply, the heat flux and the energy supply, respectively, the evolution equation for the internal energy ε^α may be expressed as

$$n^\alpha m^\alpha \varepsilon'^\alpha = n^\alpha m^\alpha r^\alpha + \boldsymbol{T}^\alpha_+ : \boldsymbol{D}^\alpha + \boldsymbol{T}^\alpha_- : \boldsymbol{W}^\alpha - \nabla \cdot \boldsymbol{q}^\alpha + \hat{\varepsilon}^\alpha , \tag{2.7}$$

where $\boldsymbol{D}^\alpha = \mathrm{sym}(\nabla \boldsymbol{v}^\alpha)$ is the stretching tensor and $\boldsymbol{W}^\alpha = \mathrm{skw}(\nabla \boldsymbol{v}^\alpha)$ is the spin tensor. The consistency of the balance of energy (2.7) with the balance of energy for the mixture requires that

$$\sum_\alpha \{\hat{\varepsilon}^\alpha + \boldsymbol{u}^\alpha \cdot \hat{\boldsymbol{p}}^\alpha + \hat{c}^\alpha[\varepsilon^\alpha + \tfrac{1}{2}(u^\alpha)^2]\} = 0 . \tag{2.8}$$

On returning now to the case of plasmas, the general balance equations (2.4), (2.7) must be supplemented with the expressions of the force $n^\alpha m^\alpha \boldsymbol{f}^\alpha$ and the heat supply $n^\alpha m^\alpha r^\alpha$; they are

$$n^\alpha m^\alpha \boldsymbol{f}^\alpha = \alpha n^\alpha e \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{v}^\alpha \times \boldsymbol{B}\right) + n^\alpha m^\alpha \boldsymbol{F}^\alpha , \tag{2.9}$$

$$n^\alpha m^\alpha r^\alpha = \alpha n^\alpha e\, \boldsymbol{v}^\alpha \cdot \boldsymbol{E} + n^\alpha m^\alpha R^\alpha , \tag{2.10}$$

the quantities $\boldsymbol{F}^\alpha$, R^α being purely mechanical in character. The electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$ are obviously subject to Maxwell's equations which, in Heaviside-Lorentz units, read

$$\nabla \times \boldsymbol{E} + \frac{1}{c} \frac{\partial \boldsymbol{B}}{\partial t} = 0 , \qquad \nabla \cdot \boldsymbol{B} = 0 , \tag{2.11}$$

$$\nabla \times \boldsymbol{B} - \frac{1}{c} \frac{\partial \boldsymbol{E}}{\partial t} = \frac{e}{c} \sum_\alpha (\alpha n^\alpha \boldsymbol{v}^\alpha) , \quad \nabla \cdot \boldsymbol{E} = e \sum_\alpha (\alpha n^\alpha) . \tag{2.12}$$

The set of fields describing the behaviour of the plasma must meet the balance equations written above. Of course, the peculiar properties of the plasma at hand are accounted for via the introduction of suitable constitutive relations; a scheme framing them in a thermodynamic theory is shown in the next section.

3. – Constitutive theory.

Standard accounts of plasmas [16] assign different temperature fields to the various constituents. Moreover, the constituents are viewed as heat-conducting viscous fluids, while the effect of collisions are represented by terms involving relative velocities of the constituents [17]. As we shall see shortly, these aspects may be embodied in a systematic scheme of plasma as a mixture of fluids with hidden variables.

Basically, a material with hidden variables [18] consists of a set of response functions

$$\varphi = \varphi(y, a)$$

and of a function h governing the evolution of the hidden variables a through the differential equation

$$\dot{a} = h(y, z, a)\,,$$

where the symbols y, z stand for suitable sets of real (physical) variables and the superposed dot denotes the relevant material time derivative.

Since the material at hand is a mixture of $\nu + 2$ fluid constituents, we assign it the structure of material with hidden variables by identifying y with the ordered array $(\theta^{-1}, \theta^0, \ldots, \theta^\nu; n^{-1}, n^0, \ldots, n^\nu; \boldsymbol{v}^{-1}, \boldsymbol{v}^0, \ldots, \boldsymbol{v}^\nu; \boldsymbol{W}^{-1}, \boldsymbol{W}^0, \ldots, \boldsymbol{W}^\nu)$ [19] and z with the ordered array $(\boldsymbol{D}^{-1}, \boldsymbol{D}^0, \ldots, \boldsymbol{D}^\nu; \boldsymbol{g}^{-1}, \boldsymbol{g}^0, \ldots, \boldsymbol{g}^\nu)$. The hidden variables a are the ordered array $(\boldsymbol{\Sigma}^{-1}, \boldsymbol{\Sigma}^0, \ldots, \boldsymbol{\Sigma}^\nu; \Theta^{-1}, \Theta^0, \ldots, \Theta^\nu; \boldsymbol{\Lambda}^{-1}, \boldsymbol{\Lambda}^0, \ldots, \boldsymbol{\Lambda}^\nu)$ of the symmetric second-order traceless tensors $\boldsymbol{\Sigma}^\alpha$, of the scalars Θ^α and of the vectors $\boldsymbol{\Lambda}^\alpha$. Finally the response functions φ are identified with the ordered array $(\psi^{-1}, \psi^0, \ldots, \psi^\nu; s^{-1}, s^0, \ldots, s^\nu; \boldsymbol{T}^{-1}, \boldsymbol{T}^0, \ldots, \boldsymbol{T}^\nu; \boldsymbol{q}^{-1}, \boldsymbol{q}^0, \ldots, \boldsymbol{q}^\nu; \hat{c}^{-1}, \hat{c}^0, \ldots, \hat{c}^\nu; \hat{\boldsymbol{p}}^{-1}, \hat{\boldsymbol{p}}^0, \ldots, \hat{\boldsymbol{p}}^\nu; \hat{\varepsilon}^{-1}, \hat{\varepsilon}^0, \ldots, \hat{\varepsilon}^\nu)$, where s^α is the entropy and $\psi^\alpha = \varepsilon^\alpha - \theta^\alpha s^\alpha$ is the free energy.

To go further we need some assumptions on the evolution function h. In principle, we could examine hypotheses general enough—see, *e.g.*, ref. [10]—; however, both to avoid inessential formal difficulties and to get a theory providing the most direct generalization of Navier-Stokes' and Fourier's laws,

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[16] See, *e.g.*, A. N. KAUFMAN: *Dissipative effects*, in *Plasma Physics in Theory and Application*, edited by W. B. KUNKEL (New York, N. Y., 1966).

[17] See, *e.g.*, P. C. CLEMMOW and J. P. DOUGHERTY: *Electrodynamics of Particles and Plasmas*, sect. **5** and **6** (Reading, Mass., 1969).

[18] The reader interested in a more refined approach to materials with hidden variables is referred to ref. [10].

[19] Of course, owing to the principle of material-frame indifference, the response functions φ and the evolution function h should depend on $\boldsymbol{v}^\alpha$ and $\boldsymbol{W}^\alpha$ only through the differences $\boldsymbol{v}^\alpha - \boldsymbol{v}^\beta$, $\boldsymbol{W}^\alpha - \boldsymbol{W}^\beta$, $\alpha, \beta = -1, 0, \ldots, \nu$.

we choose the function h so as to make, for any fixed particle $\boldsymbol{X}^\alpha$, the evolution equations into the system

$$\boldsymbol{\Sigma}^{\cdot\alpha} = \frac{1}{\tau_s^\alpha}(\langle \boldsymbol{D}^\alpha \rangle - \boldsymbol{\Sigma}^\alpha)\,, \tag{3.1}$$

$$\Theta^{\cdot\alpha} = \frac{1}{\tau_v^\alpha}(\operatorname{tr} \boldsymbol{D}^\alpha - \Theta^\alpha)\,, \tag{3.2}$$

$$\boldsymbol{\Lambda}^{\cdot\alpha} = \frac{1}{\tau_c^\alpha}(\boldsymbol{g}^\alpha - \boldsymbol{\Lambda}^\alpha)\,, \tag{3.3}$$

where $\langle \boldsymbol{D}^\alpha \rangle = \boldsymbol{D}^\alpha - \frac{1}{3}(\operatorname{tr} \boldsymbol{D}^\alpha)\boldsymbol{I}$. The positive parameters τ_s^α, τ_v^α, τ_c^α may be viewed as relaxation times.

In connection with the evolution equations (3.1)-(3.3), we observe that sometimes the material time derivative is replaced with an objective time derivative like, for example, the co-rotational one. Setting aside any question about objectivity and objective time derivatives ([20]), we remark that the results of this section are independent of the choice of the time derivative, while quantitative differences may arise in connection with wave propagation ([21]).

The set of response functions φ must satisfy the restrictions placed by the second law of thermodynamics. To make this fact operative, we take that the inequality

$$\sum_\alpha \left[s^{\cdot\alpha} + s^\alpha \hat{c}^\alpha + \nabla \cdot \left(\frac{\boldsymbol{q}^\alpha}{\theta^\alpha}\right) - \frac{\varrho^\alpha R^\alpha}{\theta^\alpha}\right] \geqslant 0$$

holds at any point of the mixture ([7]). Then, on account of (2.7), (2.10), this implies that

$$\sum_\alpha \frac{1}{\theta^\alpha}\Big[- n^\alpha m^\alpha(\psi^{\cdot\alpha} + s^\alpha \theta^{\cdot\alpha}) - \frac{1}{\theta^\alpha}\boldsymbol{q}^\alpha \cdot \boldsymbol{g}^\alpha + \boldsymbol{T}_+^\alpha : \boldsymbol{D}^\alpha + \boldsymbol{T}_-^\alpha : \boldsymbol{W}^\alpha + \hat{\varepsilon}^\alpha + \\ + \theta^\alpha s^\alpha \hat{c}^\alpha + \alpha e n^\alpha \boldsymbol{v}^\alpha \cdot \boldsymbol{E}\Big] \geqslant 0 \tag{3.4}$$

must be true identically. According to our constitutive assumptions, the free energy ψ^α of the α-th constituent is given by a function of the form

$$\psi^\alpha = \psi^\alpha(\theta^{-1}, \dots, \theta^\nu; n^{-1}, \dots, n^\nu; \boldsymbol{v}^{-1}, \dots, \boldsymbol{v}^\nu; \boldsymbol{W}^{-1}, \dots, \boldsymbol{W}^\nu; \\ \boldsymbol{\Sigma}^{-1}, \dots, \boldsymbol{\Sigma}^\nu; \Theta^{-1}, \dots, \Theta^\nu; \boldsymbol{\Lambda}^{-1}, \dots, \boldsymbol{\Lambda}^\nu):$$

Hence, in view of the evolution equations (3.1)-(3.3),the entropy inequality (3.4)

([20]) See, *e.g.*, F. BAMPI and A. MORRO: *Found. Phys.*, **10**, 905 (1980).
([21]) F. BAMPI and A. MORRO: *J. Phys. A*, **14**, 631 (1981).

may be expressed as

$$(3.5)\qquad \sum_\alpha \sum_\beta \frac{1}{\theta^\alpha}\Big\{-n^\alpha m^\alpha(\psi^\alpha_{\theta^\beta}+s^\alpha\delta^{\alpha\beta})\,\theta'^\beta+\Big(\boldsymbol{T}^\alpha_+\delta^{\alpha\beta}-\frac{n^\alpha m^\alpha}{\tau^\beta_s}\psi^\alpha_{\boldsymbol{\Sigma}^\beta}-\frac{n^\alpha m^\alpha}{\tau^\beta_v}\psi^\alpha_{\Theta^\beta}\boldsymbol{I}+$$
$$+n^\beta n^\alpha m^\alpha\psi^\alpha_{n^\beta}\boldsymbol{I}\Big):\boldsymbol{D}^\beta-\Big(\frac{1}{\theta^\alpha}\,\boldsymbol{q}^\alpha\delta^{\alpha\beta}+\frac{n^\alpha m^\alpha}{\tau^\beta_c}\psi^\alpha_{\boldsymbol{\Lambda}^\beta}\Big)\cdot\boldsymbol{g}^\beta-$$
$$-n^\alpha m^\alpha\psi^\alpha_{\boldsymbol{v}^\beta}\cdot\boldsymbol{v}'^\beta-n^\alpha m^\alpha\psi^\alpha_{\boldsymbol{W}^\beta}:\boldsymbol{W}'^\beta+\frac{n^\alpha m^\alpha}{\tau^\beta_s}\psi^\alpha_{\boldsymbol{\Sigma}^\beta}:\boldsymbol{\Sigma}^\beta+\frac{n^\alpha m^\alpha}{\tau^\beta_v}\psi^\alpha_{\Theta^\beta}\Theta^\beta+\frac{n^\alpha m^\alpha}{\tau^\beta_c}\psi^\alpha_{\boldsymbol{\Lambda}^\beta}\cdot\boldsymbol{\Lambda}^\beta+$$
$$+(\boldsymbol{T}^\alpha_-:\boldsymbol{W}^\alpha+\hat{\varepsilon}^\alpha+\alpha e n^\alpha\boldsymbol{v}^\alpha\cdot\boldsymbol{E})\,\delta^{\alpha\beta}+\Big(\theta^\alpha s^\alpha\delta^{\alpha\beta}-\frac{n^\alpha m^\alpha}{m^\beta}\psi^\alpha_{n^\beta}\Big)\hat{c}^\beta-$$
$$-n^\alpha m^\alpha\psi^\alpha_{\theta^\beta}(\boldsymbol{v}^\alpha-\boldsymbol{v}^\beta)\cdot\boldsymbol{g}^\beta-n^\alpha m^\alpha\psi^\alpha_{n^\beta}(\boldsymbol{v}^\alpha-\boldsymbol{v}^\beta)\cdot\nabla n^\beta-$$
$$-n^\alpha m^\alpha\psi^\alpha_{\boldsymbol{v}^\beta}\cdot[(\boldsymbol{v}^\alpha-\boldsymbol{v}^\beta)\cdot\nabla]\boldsymbol{v}^\beta-n^\alpha m^\alpha\psi^\alpha_{\boldsymbol{W}^\beta}:[(\boldsymbol{v}^\alpha-\boldsymbol{v}^\beta)\cdot\nabla]\boldsymbol{W}^\beta-$$
$$-n^\alpha m^\alpha\psi^\alpha_{\boldsymbol{\Sigma}^\beta}:[(\boldsymbol{v}^\alpha-\boldsymbol{v}^\beta)\cdot\nabla]\boldsymbol{\Sigma}^\beta-n^\alpha m^\alpha\psi^\alpha_{\Theta^\beta}(\boldsymbol{v}^\alpha-\boldsymbol{v}^\beta)\cdot\nabla\Theta^\beta-$$
$$-n^\alpha m^\alpha\psi^\alpha_{\boldsymbol{\Lambda}^\beta}\cdot[(\boldsymbol{v}^\alpha-\boldsymbol{v}^\beta)\cdot\nabla]\boldsymbol{\Lambda}^\beta\Big\}\geqslant 0\,,$$

where the subscripts θ^β, n^β, $\boldsymbol{v}^\beta$, $\boldsymbol{W}^\beta$, $\boldsymbol{\Sigma}^\beta$, Θ^β, $\boldsymbol{\Lambda}^\beta$ denote partial differentiations and $\delta^{\alpha\beta}$ is the usual Krönecker delta. Our purpose is now to exploit inequality (3.5) so as to derive the main restrictions placed by the second law of thermodynamics on our constitutive assumptions. To do this, we recall first that, as always, the present values of the hidden variables $\alpha(t)$ are independent of the present values of the real variables $z(t)$ [9,10]. Accordingly, the quantities $\boldsymbol{v}'^\beta$ and $\boldsymbol{W}'^\beta$ can be chosen arbitrarily [22] and independently of the other variables appearing in (3.5). Therefore, inequality (3.5) holds identically only if

$$(3.6)\qquad \sum_\alpha \frac{n^\alpha m^\alpha}{\theta^\alpha}\psi^\alpha_{\boldsymbol{v}^\beta}=0\,,\qquad \sum_\alpha \frac{n^\alpha m^\alpha}{\theta^\alpha}\psi^\alpha_{\boldsymbol{W}^\beta}=0\,.$$

Moreover, since $\boldsymbol{\Sigma}^\alpha(t)$, $\Theta^\alpha(t)$ and $\boldsymbol{\Lambda}^\alpha(t)$ are independent of $\boldsymbol{D}^\alpha(t)$, $\boldsymbol{g}^\alpha(t)$ and $\theta'^\beta(t)$, $\boldsymbol{D}^\beta(t)$, $\boldsymbol{g}^\beta(t)$ are arbitrary, inequality (3.5) implies that

$$(3.7)\qquad s^\beta=-\frac{\theta^\beta}{n^\beta m^\beta}\sum_\alpha\frac{n^\alpha m^\alpha}{\theta^\alpha}\psi^\alpha_{\theta^\beta}\,,$$

$$(3.8)\qquad \boldsymbol{T}^\beta_+=\theta^\beta\sum_\alpha\frac{1}{\theta^\alpha}\Big[-n^\beta n^\alpha m^\alpha\psi^\alpha_{n^\beta}\boldsymbol{I}+\frac{n^\alpha m^\alpha}{\tau^\beta_s}\psi^\alpha_{\boldsymbol{\Sigma}^\beta}+\frac{n^\alpha m^\alpha}{\tau^\beta_v}\psi^\alpha_{\Theta^\beta}\boldsymbol{I}+$$
$$+\frac{1}{2}n^\alpha m^\alpha(\boldsymbol{v}^\alpha-\boldsymbol{v}^\beta)\otimes\psi^\alpha_{\boldsymbol{v}^\beta}+\frac{1}{2}n^\alpha m^\alpha\psi^\alpha_{\boldsymbol{v}^\beta}\otimes(\boldsymbol{v}^\alpha-\boldsymbol{v}^\beta)\Big]\,,$$

$$(3.9)\qquad \boldsymbol{q}^\beta=-(\theta^\beta)^2\sum_\alpha\frac{1}{\theta^\alpha}\Big[\frac{n^\alpha m^\alpha}{\tau^\beta_c}\psi^\alpha_{\boldsymbol{\Lambda}^\beta}+n^\alpha m^\alpha\psi^\alpha_{\theta^\beta}(\boldsymbol{v}^\alpha-\boldsymbol{v}^\beta)\Big]\,.$$

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[22] On appealing to the arbitrariness of the external-body forces $n^\beta m^\beta \boldsymbol{F}^\beta$ occurring in eq. (2.4), the accelerations $\boldsymbol{v}'^\beta$ may be viewed as arbitrary vectors.

Now, appeal to the arbitrariness of the present values $\nabla\boldsymbol{W}^\beta$, $\nabla\boldsymbol{\Sigma}^\beta$, $\nabla\Theta^\beta$, $\nabla\boldsymbol{\Lambda}^\beta$ leads to

$$\sum_\alpha \frac{n^\alpha m^\alpha}{\theta^\alpha}(\boldsymbol{v}^\alpha - \boldsymbol{v}^\beta)\otimes \psi^{\alpha}_{\boldsymbol{W}^\beta} = 0\,, \tag{3.10}$$

$$\sum_\alpha \frac{n^\alpha m^\alpha}{\theta^\alpha}(\boldsymbol{v}^\alpha - \boldsymbol{v}^\beta)\otimes \psi^{\alpha}_{\boldsymbol{\Sigma}^\beta} = 0\,, \tag{3.11}$$

$$\sum_\alpha \frac{n^\alpha m^\alpha}{\theta^\alpha}(\boldsymbol{v}^\alpha - \boldsymbol{v}^\beta)\, \psi^{\alpha}_{\Theta^\beta} = 0\,, \tag{3.12}$$

$$\sum_\alpha \frac{n^\alpha m^\alpha}{\theta^\alpha}(\boldsymbol{v}^\alpha - \boldsymbol{v}^\beta)\otimes \psi^{\alpha}_{\boldsymbol{\Lambda}^\beta} = 0\,. \tag{3.13}$$

Finally, upon substituting these conditions into inequality (3.5), we find straightaway the reduced dissipation inequality

$$\begin{aligned}\sum_\alpha\sum_\beta \frac{1}{\theta^\alpha}\Bigg[&\frac{n^\alpha m^\alpha}{\tau^\beta_s}\psi^{\alpha}_{\boldsymbol{\Sigma}^\beta}:\boldsymbol{\Sigma}^\beta + \frac{n^\alpha m^\alpha}{\tau^\beta_v}\psi^{\alpha}_{\Theta^\beta}\Theta^\beta + \frac{n^\alpha m^\alpha}{\tau^\beta_c}\psi^{\alpha}_{\boldsymbol{\Lambda}^\beta}\cdot\boldsymbol{\Lambda}^\beta + \\ &+ (\boldsymbol{T}^\alpha_-:\boldsymbol{W}^\alpha + \hat{\varepsilon}^\alpha + \alpha e n^\alpha \boldsymbol{v}^\alpha\cdot\boldsymbol{E})\,\delta^{\alpha\beta} - n^\alpha m^\alpha(\boldsymbol{v}^\alpha - \boldsymbol{v}^\beta)\cdot(\boldsymbol{W}^\beta \zeta^{\alpha}_{\boldsymbol{v}^\beta}) - \\ &- n^\alpha m^\alpha(\boldsymbol{v}^\alpha - \boldsymbol{v}^\beta)\cdot\nabla n^\beta \psi^{\alpha}_{n^\beta} - \frac{n^\alpha m^\alpha}{m^\beta}\hat{c}^\beta\left(\frac{\theta^\beta}{n^\beta}\zeta^{\alpha}_{\theta^\beta} + \zeta^{\alpha}_{n^\beta}\right)\Bigg] \geqslant 0\,.\end{aligned} \tag{3.14}$$

Thus we may summarize the results so obtained as follows.

Look at a plasma as a mixture of $\nu + 2$ fluid constituents with hidden variables whose behaviour is accounted for through the constitutive assumptions stated above. Then conditions (3.6)-(3.14) are necessary and sufficient for the identical validity of inequality (3.5). Indeed, the *only if* part has been proved, while the *if* part is obvious.

We end this section with the following remarks.

Remark 1. A doubt could be cast upon inserting the spin tensors $\boldsymbol{W}^\alpha$ in the y-like variables instead of into z-like ones. The insertion of $\boldsymbol{W}^\alpha$ into z, or possibly into the evolution equations for the hidden variables, should be suggested by physical arguments. It seems that suggestions in this sense are not available. Looking at $\boldsymbol{W}^\alpha$ as y-like variable allows $\boldsymbol{T}^\alpha_-$ to be compatible with thermodynamics without being identically vanishing and, meanwhile, it permits us to account for a dependence on $\boldsymbol{W}^\alpha$.

Remark 2. The fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are determined by the quantities $\sum_\alpha \alpha n^\alpha$ $\sum_\alpha \alpha n^\alpha \boldsymbol{v}^\alpha$ through the Maxwell equations (2.11), (2.12). Nevertheless, we could consider response functions φ dependent on the values of $\boldsymbol{E}$ and $\boldsymbol{B}$ as well.

Owing to lack of physical motivation and in order to avoid inessential formal complications, here we do not account for such a dependence which will be treated again in sect. **6**.

4. – A description of dissipative phenomena in plasmas.

In view of eqs. (3.7)-(3.9), any set of free energy functions ψ^α's meeting inequality (3.14) provides response functions s^α, $\boldsymbol{T}_+^\alpha$, $\boldsymbol{q}^\alpha$ automatically consistent with the second law of thermodynamics in the form of inequality (3.4). Of course, on the basis of a merely mathematical standpoint, we cannot prefer any set of functions ψ^α among all those compatible with (3.14). An important special case compatible with (3.14) is delivered by the ideal-fluid mixture approximation (7) in which the free energy ψ^α of the α-th constituent depends on densities and temperatures only through ϱ^α and θ^α. Beside this approximation, physical arguments allow us to select a particular class of free energy functions in that, for any constituent, the ansatz

$$\psi^\alpha = \Psi^\alpha(\theta^\alpha, n^\alpha) + \frac{1}{n^\alpha m^\alpha}\left[\eta^\alpha \tau_s^\alpha \boldsymbol{\Sigma}^\alpha : \boldsymbol{\Sigma}^\alpha + \frac{1}{2}\zeta^\alpha \tau_v^\alpha (\Theta^\alpha)^2 + \frac{\varkappa^\alpha \tau_c^\alpha}{2\theta^\alpha} \boldsymbol{\Lambda}^\alpha \cdot \boldsymbol{\Lambda}^\alpha\right], \tag{4.1}$$

η^α, ζ^α, $\varkappa^\alpha$ being real constants, leads to the most natural generalization of Navier-Stokes' and Fourier's laws. To make this assertion precise, observe first that substitution in (3.7)-(3.9) yields the response functions

$$s^\alpha = -\Psi^\alpha_{\theta^\alpha} + \frac{\varkappa^\alpha \tau_c^\alpha}{(2\theta^\alpha)^2} \boldsymbol{\Lambda}^\alpha \cdot \boldsymbol{\Lambda}^\alpha , \tag{4.2}$$

$$\boldsymbol{T}_+^\alpha = -p^\alpha \boldsymbol{I} + 2\eta^\alpha \boldsymbol{\Sigma}^\alpha + \zeta^\alpha \Theta^\alpha \boldsymbol{I} , \tag{4.3}$$

$$\boldsymbol{q}^\alpha = -\varkappa^\alpha \boldsymbol{\Lambda}^\alpha , \tag{4.4}$$

where $p^\alpha = (n^\alpha)^2 m^\alpha \psi^\alpha_{n^\alpha}$ is the pressure of the α-th constituent. Now the reduced dissipation inequality (3.14) must hold identically. In particular, it must be true in the case of a process such that, at any time t, $\boldsymbol{E} = 0$, $\boldsymbol{W}^\alpha = 0$, $\hat{c}^\alpha = 0$, $\hat{\varepsilon}^\alpha = 0$. In this instance, since the hidden variables are independent of each other, we achieve the $\nu + 2$ partial inequalities

$$\frac{1}{\tau_s^\alpha} \psi^\alpha_{\boldsymbol{\Sigma}^\alpha} : \boldsymbol{\Sigma}^\alpha + \frac{1}{\tau_v^\alpha} \psi^\alpha_{\Theta^\alpha} \Theta^\alpha + \frac{1}{\tau_c^\alpha} \psi^\alpha_{\boldsymbol{\Lambda}^\alpha} \cdot \boldsymbol{\Lambda}^\alpha \geqslant 0 ,$$

whence

$$\eta^\alpha \geqslant 0 , \qquad \zeta^\alpha \geqslant 0 , \qquad \varkappa^\alpha \geqslant 0 . \tag{4.5}$$

The meaning of the parameters η^α, ζ^α, $\varkappa^\alpha$ arises from an analysis of stationary processes characterized by the constancy in time of $\boldsymbol{g}^\alpha$ and $\boldsymbol{D}^\alpha$. In such a case asymptotically we have

$$\boldsymbol{\Sigma}^\alpha = \langle \boldsymbol{D}^\alpha \rangle\,, \qquad \Theta^\alpha = \operatorname{tr} \boldsymbol{D}^\alpha\,, \qquad \boldsymbol{\Lambda}^\alpha = \boldsymbol{g}^\alpha\,,$$

and hence eqs. (4.3), (4.4) become the standard laws of viscosity and heat conduction. This enables us to identify the parameters η^α, ζ^α, $\varkappa^\alpha$ with the usual viscosity and heat conduction coefficients and to regard the restrictions (4.5) as Stokes-Duhem's and Fourier's inequalities.

Remark 3. The explicit form of the partial pressure p^α, namely

$$p^\alpha = (n^\alpha)^2 m^\alpha \Psi^\alpha_{n^\alpha} - \left[\eta^\alpha \tau^\alpha_s \boldsymbol{\Sigma}^\alpha : \boldsymbol{\Sigma}^\alpha + \frac{1}{2}\zeta^\alpha \tau^\alpha_v (\Theta^\alpha)^2 + \frac{\varkappa^\alpha \tau^\alpha_c}{2\theta^\alpha}\boldsymbol{\Lambda}^\alpha \cdot \boldsymbol{\Lambda}^\alpha\right],$$

shows that the actual pressure differs from the hydrostatic pressure $(n^\alpha)^2 m^\alpha \Psi^\alpha_{n^\alpha}$ as to a quadratic contribution of the hidden variables. Accordingly, eq. (4.3) reduces exactly to Navier-Stokes' equation in the linear approximation.

5. – Energy and momentum transfer between constituents.

The literature on plasmas bears evidence of particular expressions for energy and momentum transfer between ions and electrons. Such expressions are usually suggested by kinetic-theory arguments; hence it is of interest to examine whether and how they may be embodied into the present scheme of plasmas as mixtures.

Look at nonreacting constituents, *i.e.* $\hat{c}^\alpha = 0$, in the case in which the partial stresses are symmetric, *i.e.* $\boldsymbol{T}^\alpha_- = 0$. The interactions between constituents then show up as the growths of energy l^α and the growths of linear momentum $\boldsymbol{m}^\alpha$ (see appendix A). As usual, we take that $\boldsymbol{m}^\alpha$, namely the momentum transferred to the α-th constituent by collisions with all the other constituents, may be written as [6]

$$\boldsymbol{m}^\alpha = \sum_\beta M^{\alpha\beta}(\boldsymbol{v}^\beta - \boldsymbol{v}^\alpha) \tag{5.1}$$

where

$$\sum_\alpha M^{\alpha\beta} - M^{\beta\alpha} = 0\,, \tag{5.2}$$

in view of requirement (A.5). Meanwhile, it is reasonable to assume that the growths of energy l^α involve both differences of temperature and differences of kinetic energy. Accordingly, set

$$l^\alpha = \sum_\beta \{K^{\alpha\beta}(\theta^\beta - \theta^\alpha) + \tfrac{1}{2}N^{\alpha\beta}[(v^\beta)^2 - (v^\alpha)^2]\}\,; \tag{5.3}$$

condition (A.6) requires that

$$\sum_\alpha K^{\alpha\beta} - K^{\beta\alpha} = 0\,, \qquad \sum_\alpha N^{\alpha\beta} - N^{\beta\alpha} = 0\,; \tag{5.4}$$

henceforth $K^{\alpha\beta}$, $M^{\alpha\beta}$ and $N^{\alpha\beta}$ are assumed to be symmetric.

Consider now the entropy inequality (3.14). The independence of the growths $\boldsymbol{m}^\alpha$ and l^α, and hence $\hat{\varepsilon}^\alpha$, of the hidden variables $\boldsymbol{\Sigma}^\alpha$, Θ^α and $\boldsymbol{\Lambda}^\alpha$, together with the arbitrariness of the electric field $\boldsymbol{E}$, the spin tensors $\boldsymbol{W}^\alpha$ and the density gradients $\boldsymbol{\xi}^\alpha$, allows us to write

$$\sum_\alpha \hat{\varepsilon}^\alpha / \theta^\alpha \geqslant 0\,. \tag{5.5}$$

Thus, as shown in appendix B, in the case of constituents at different temperatures, assumptions (5.1), (5.3) are compatible with thermodynamics if and only if ([23])

$$N^{\alpha\beta} = M^{\alpha\beta}\,, \qquad \alpha \neq \beta, \tag{5.6}$$

$$\sum_\alpha M^{\alpha\beta} \geqslant 0\,, \qquad \sum_\alpha K^{\alpha\beta} \geqslant 0\,, \qquad \alpha \neq \beta. \tag{5.7}$$

Some comments seem to be in order. The contribution of the temperature differences to the growths of energy has already been investigated in ref. ([5]), where the positiveness of the quantity $K^{\alpha\beta}$ is proved via kinetic-theory arguments. Similarly, kinetic theory accounts for the positiveness of the quantity $M^{\alpha\beta}$. Thus the inequalities (5.7) merely restate known properties. Then the new result emerging from this section is that, in view of compatibility with thermodynamics, growths of energy and growths of linear momentum are interrelated through conditions (5.6). Thus the presence of growths of linear momentum implies the presence of growths of energy. If, however, the constituents are at a common temperature, relation (5.6) is no longer a consequence of thermodynamics; that is why usually $N^{\alpha\beta}$ is assumed to vanish.

6. – From plasma physics to magnetofluidodynamics.

Magnetofluidodynamics deals with the motion of conducting fluids. As such, it may be viewed as limit of plasma physics under the approximation of high density. On the other hand, conducting fluids, or dense ionized gases, are characterized by a high collision frequency. In this regime, under the action of applied fields, electrons and ions move in such a way that there is no separation of charge $\left(i.e.\ \sum_\alpha \alpha n^\alpha \simeq 0\right)$. Basically we may summarize the

([23]) We remind the reader that the diagonal terms of the matrices N, M, K have no physical meaning.

effects of high collision frequency by saying that the motions and the temperatures of electrons, ions and atoms nearly coincide (*i.e.* $\boldsymbol{u}^\alpha \simeq 0$, $\theta^\alpha \simeq \theta$), while the net effect of ionization vanishes (*i.e.* $\hat{c}^\alpha \simeq 0$) and the densities of the constituents are subject to the condition [24]

$$n \equiv n^e \simeq n^i \ll n^a\,, \qquad n^\alpha \simeq 0, \qquad \alpha \geqslant 2\,. \tag{6.1}$$

On the basis of these observations, we move on to set up suitable equations for magnetofluidodynamics starting from equations of plasmas as mixtures of fluids. Consider first the continuity equation (2.1). If we let $\varrho = \sum_\alpha n^\alpha m^\alpha$, summation over α gives

$$\dot{\varrho} + \varrho \nabla\cdot\boldsymbol{v} = 0\,, \tag{6.2}$$

where $\boldsymbol{v} = \sum_\alpha n^\alpha m^\alpha \boldsymbol{v}^\alpha/\varrho$ is the mean velocity and the superposed dot stands for the corresponding material time derivative. According to the above considerations, henceforth we approximate the mass density $n^a m^a$ and the velocity $\boldsymbol{v}^a$ with ϱ and $\boldsymbol{v}$. Look now at the balance of linear momentum (2.4) and sum over α. On account of (2.5), (2.9), we find that

$$\varrho\dot{\boldsymbol{v}} = \nabla\cdot\boldsymbol{T} + \boldsymbol{J}\times\boldsymbol{B} + \varrho\boldsymbol{F}\,, \tag{6.3}$$

where $\boldsymbol{T} = \sum_\alpha \boldsymbol{T}^\alpha$ is the total symmetric stress and $\boldsymbol{J} = e\sum_\alpha n^\alpha \boldsymbol{v}^\alpha$ is the electric current. It is worth remarking that, besides occurring in eq. (6.3), the electric current $\boldsymbol{J}$ satisfies a generalized Ohm's law which extends Spitzer's one [4] to the case in which neutral atoms are present. To this end, we observe that multiplication of eq. (2.4) by $\alpha e/m^\alpha$ and summation over α provide

$$ne(\boldsymbol{v}'^i - \boldsymbol{v}'^e) = e\nabla\cdot\left(\frac{\boldsymbol{T}^i}{m^i} - \frac{\boldsymbol{T}^e}{m^e}\right) + e\left(\frac{\hat{\boldsymbol{p}}^i}{m^i} - \frac{\hat{\boldsymbol{p}}^e}{m^e}\right) + \\ + ne^2\left(\frac{1}{m^i} - \frac{1}{m^e}\right)\boldsymbol{E} + \frac{ne^2}{c}\left(\frac{\boldsymbol{v}^i}{m^i} + \frac{\boldsymbol{v}^e}{m^e}\right)\times\boldsymbol{B}\,.$$

Moreover, express the forces $\hat{\boldsymbol{p}}^i$ and $\hat{\boldsymbol{p}}^e$ via eq. (5.1), namely

$$\hat{\boldsymbol{p}}^i = M^{ie}(\boldsymbol{v}^e - \boldsymbol{v}^i) + M^{ia}(\boldsymbol{v}^a - \boldsymbol{v}^i)\,, \qquad \hat{\boldsymbol{p}}^e = M^{ie}(\boldsymbol{v}^i - \boldsymbol{v}^e) + M^{ea}(\boldsymbol{v}^a - \boldsymbol{v}^e)\,.$$

Therefore, on assuming that the coefficients M^{ia}, M^{ea} are simply proportional

[24] To avoid misunderstandings, in this section we denote the constituents by the indices e (electrons), a (atoms), i (ions) instead of -1, 0, 1.

to the masses m^i and m^e, namely $M^{ia} = m^i\chi$, $M^{ea} = m^e\chi$, and on accounting for the identity

$$n\left(\frac{\boldsymbol{v}^i}{m^i} - \frac{\boldsymbol{v}^e}{m^e}\right) = \frac{n}{m^i m^e}(m^i\boldsymbol{v}^i + m^e\boldsymbol{v}^e) + \frac{1}{e}\frac{m^e - m^i}{m^e m^i}\boldsymbol{J},$$

in the linear approximation we find that

$$\text{(6.4)}\qquad n\dot{\boldsymbol{J}} = ne^2\frac{m^i + m^e}{m^i m^e}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right) + \left(M^{ie}\frac{m^i + m^e}{m^i m^e} + \chi\right)\boldsymbol{J} + \frac{ne}{c}\frac{m^e - m^i}{m^i m^e}\boldsymbol{J}\times\boldsymbol{B} + e\nabla\cdot\left(\frac{\boldsymbol{T}^i}{m^i} - \frac{\boldsymbol{T}^e}{m^e}\right),$$

since $m^i\boldsymbol{v}^i + m^e\boldsymbol{v}^e \simeq (m^i + m^e)\boldsymbol{v}$.

As to the balance of energy, we emphasize that, within the approximations stated at the beginning of the section, we have $\boldsymbol{D}^\alpha \simeq \boldsymbol{D}^a \equiv \boldsymbol{D}$, $\boldsymbol{W}^\alpha \simeq \boldsymbol{W}^a \equiv \boldsymbol{W}$ for every α, while eq. (2.8) reads $\sum_\alpha \hat{\varepsilon}^\alpha = 0$. Thus the summation over α of eqs. (2.7) and the use of (2.6), (2.10) yield

$$\text{(6.5)}\qquad \varrho\dot{\varepsilon} = \varrho R + \boldsymbol{J}\cdot\boldsymbol{E} + \boldsymbol{T}:\boldsymbol{D} - \nabla\cdot\boldsymbol{q},$$

where $\varepsilon = \sum_\alpha n^\alpha m^\alpha \varepsilon^\alpha/\varrho \simeq \varepsilon^a$ is the internal energy and $\boldsymbol{q} = \sum_\alpha \boldsymbol{q}^\alpha$ is the heat flux. Finally the entropy inequality (3.4) takes the form

$$\text{(6.6)}\qquad -\varrho(\dot{\psi} + s\dot{\theta}) - \frac{1}{\theta}\boldsymbol{g}\cdot\boldsymbol{q} + \boldsymbol{T}:\boldsymbol{D} + \boldsymbol{J}\cdot\boldsymbol{E} \geqslant 0,$$

since $\psi = \sum_\alpha n^\alpha m^\alpha \psi^\alpha/\varrho \simeq \psi^a$ and $s = \sum_\alpha n^\alpha m^\alpha s^\alpha/\varrho \simeq s^a$.

Equations (6.2), (6.3), (6.5), (6.6) are exactly the standard balance equations of magnetofluidodynamics. Customarily these equations are supplemented by a constitutive relation for the electric current $\boldsymbol{J}$. In the present context eq. (6.4) provides a natural suggestion for such a relation; a comprehensive scheme of magnetofluidodynamics which embodies a constitutive equation of this type is delivered in ref. ([14]).

* * *

The authors wish to thank Prof. I. MÜLLER for fruitful discussions about some aspects of balance equations for mixtures. This work was supported by CNR through the project « Conservazione del Suolo », subproject « Dinamica dei Litorali ».

Appendix A

The balance equations governing the behaviour of a reacting mixture may be derived straightaway from their intergral version—see, *e.g.*, ref. [25]. In so doing, the balance equations for mass, linear momentum and energy take the form

$$\text{(A.1)} \qquad m^\alpha\left[\frac{\partial n^\alpha}{\partial t} + \nabla\cdot(n^\alpha \boldsymbol{v}^\alpha)\right] = \hat{c}^\alpha ,$$

$$\text{(A.2)} \qquad \frac{\partial}{\partial t}(n^\alpha m^\alpha \boldsymbol{v}^\alpha) + \nabla\cdot(n^\alpha m^\alpha \boldsymbol{v}^\alpha \otimes \boldsymbol{v}^\alpha - \boldsymbol{T}^\alpha) - n^\alpha m^\alpha \boldsymbol{f}^\alpha = \boldsymbol{m}^\alpha ,$$

$$\text{(A.3)} \qquad \frac{\partial}{\partial t}\left[n^\alpha m^\alpha\left(\varepsilon^\alpha + \frac{(v^\alpha)^2}{2}\right)\right] + \nabla\cdot\left[n^\alpha m^\alpha\left(\varepsilon^\alpha + \frac{(v^\alpha)^2}{2}\right)\boldsymbol{v}^\alpha - (\boldsymbol{T}^\alpha)^{\mathrm{T}}\boldsymbol{v}^\alpha + \boldsymbol{q}^\alpha\right] - \\ - n^\alpha m^\alpha(\boldsymbol{f}^\alpha\cdot\boldsymbol{v}^\alpha + r^\alpha) = l^\alpha ,$$

the quantities $\hat{c}^\alpha$, $\boldsymbol{m}^\alpha$, l^α being called growth of mass, growth of linear momentum and growth of energy, respectively. The requirement that mass, linear momentum and energy are conserved for the mixture as a whole implies that

$$\text{(A.4)} \qquad \sum_\alpha \hat{c}^\alpha = 0 ,$$

$$\text{(A.5)} \qquad \sum_\alpha \boldsymbol{m}^\alpha = 0 ,$$

$$\text{(A.6)} \qquad \sum_\alpha l^\alpha = 0 .$$

Equations (2.1), (2.4) and (2.7) of the text are recovered from eqs. (A.1)-(A.3) through direct calculation. In particular, it results that the growths of mass are just the mass supplies, while

$$\text{(A.7)} \qquad \hat{\boldsymbol{p}}^\alpha = \boldsymbol{m}^\alpha - c^\alpha \boldsymbol{v}^\alpha ,$$

$$\text{(A.8)} \qquad \hat{\varepsilon}^\alpha = l^\alpha - \boldsymbol{m}^\alpha\cdot\boldsymbol{v}^\alpha - \hat{c}^\alpha\left(\varepsilon^\alpha - \frac{(v^\alpha)^2}{2}\right).$$

In view of relations (A.5), (A.8), conditions (2.5)-(2.8) are completely equivalent to properties (A.4)-(A.6).

Appendix B

Assume the quantities $\boldsymbol{m}^\alpha$ and l^α to be given by eqs. (5.1), (5.3). On account of (A.8), we get

$$\sum_\alpha \frac{\hat{\varepsilon}^\alpha}{\theta^\alpha} = \sum_\alpha\sum_\beta \frac{1}{\theta^\alpha}\left\{K^{\alpha\beta}(\theta^\beta - \theta^\alpha) + \frac{1}{2}N^{\alpha\beta}\left[(v^\beta)^2 - (v^\alpha)^2\right] - M^{\alpha\beta}(\boldsymbol{v}^\beta - \boldsymbol{v}^\alpha)\cdot\boldsymbol{v}^\alpha\right\}.$$

[25] C. A. Truesdell: *Rational Thermodynamics* (New York, N. Y., 1969).

Symmetrize this expression with respect to α, β by interchanging α, β and by adding the resulting expressions together. In view of eqs. (5.2), (5.4) we have

$$\sum_\alpha \frac{\hat{\varepsilon}^\alpha}{\theta^\alpha} = \frac{1}{2}\sum_\alpha \sum_\beta \left\{K^{\alpha\beta}(\theta^\alpha - \theta^\beta)\left(\frac{1}{\theta^\alpha} - \frac{1}{\theta^\beta}\right) + \frac{1}{2} N^{\alpha\beta}[(v^\beta)^2 - (v^\alpha)^2]\left(\frac{1}{\theta^\alpha} - \frac{1}{\theta^\beta}\right) + \right.$$
$$\left. + M^{\alpha\beta}(\boldsymbol{v}^\alpha - \boldsymbol{v}^\beta)\cdot\left(\frac{\boldsymbol{v}^\alpha}{\theta^\alpha} - \frac{\boldsymbol{v}^\beta}{\theta^\beta}\right)\right\}.$$

Now, owing to the identity

$$(\boldsymbol{v}^\alpha - \boldsymbol{v}^\beta)\cdot\left(\frac{\boldsymbol{v}^\alpha}{\theta^\alpha} - \frac{\boldsymbol{v}^\beta}{\theta^\beta}\right) = \frac{1}{2}\left(\frac{1}{\theta^\alpha} + \frac{1}{\theta^\beta}\right)(\boldsymbol{v}^\alpha - \boldsymbol{v}^\beta)^2 + \frac{1}{2}[(v^\alpha)^2 - (v^\beta)^2]\left(\frac{1}{\theta^\alpha} - \frac{1}{\theta^\beta}\right),$$

we obtain

$$\sum_\alpha \frac{\hat{\varepsilon}^\alpha}{\theta^\alpha} = \frac{1}{2}\sum_\alpha \sum_\beta \left\{K^{\alpha\beta}\frac{(\theta^\alpha - \theta^\beta)^2}{\theta^\alpha\theta^\beta} + \frac{1}{2} M^{\alpha\beta}\left(\frac{1}{\theta^\alpha} + \frac{1}{\theta^\beta}\right)(\boldsymbol{v}^\alpha - \boldsymbol{v}^\beta)^2 + \right.$$
$$\left. + \frac{1}{2}(N^{\alpha\beta} - M^{\alpha\beta})[(v^\beta)^2 - (v^\alpha)^2]\left(\frac{1}{\theta^\alpha} - \frac{1}{\theta^\beta}\right)\right\}.$$

Thus inequality (5.5) provides [23]

$$\sum_\alpha K^{\alpha\beta} \geqslant 0\,, \qquad \sum_\alpha M^{\alpha\beta} \geqslant 0\,, \qquad \alpha \neq \beta\,.$$

Also, on applying the Galilean transformation $\boldsymbol{v}^\beta \to \boldsymbol{v}^\beta + \boldsymbol{c}$ and appealing to the arbitrariness of $\boldsymbol{v}^\beta - \boldsymbol{v}^\nu$, $\theta^\beta - \theta^\nu$, it follows that $N^{\alpha\beta} = M^{\alpha\beta}$ $(\alpha \neq \beta)$ if the constituents are at different temperatures, while $N^{\alpha\beta}$ is completely unrestricted if the constituents are at a common temperature.

● RIASSUNTO

Sulla base della teoria delle misture elaborata negli ultimi anni si trattano i plasmi in maniera sistematica come misture di costituenti fluidi carichi interagenti. La viscosità e la conduzione di calore sono descritti mediante il formalismo delle variabili nascoste in modo da rendere ammissibile la propagazione di fronti d'onda. L'analisi della compatibilità del modello con la termodinamica è effettuata considerando la seconda legge nella forma della diseguaglianza di Clausius-Duhem; come risultato si ottiene uno schema completo di effetti dissipativi nei plasmi. Per esempio, si mostra che nel caso di costituenti a temperatura diversa un trasferimento d'impulso è inevitabilmente legato ad un corrispondente trasferimento di energia. Infine si mostra che, nel caso limite della magnetofluidodinamica, il fenomeno della conduzione elettrica può essere inglobato nel contesto degli effetti dissipativi.

Подход, использующий теорию смесей, к плазме с диссипативными явлениями.

Резюме (*). — На основе недавно развитой теории смесей описывается плазма систематическим образом, как смесь взаимодействующих заряженных жидких компонент. Вязкость и теплопроводность этих компонент объясняются с помощью формализма скрытых переменных, что позволяет описать распространение волнового фронта. Проводится анализ совместимости предложенной модели со вторым законом термодинамики в форме неравенства Клаузиуса-Духема. В результате этого получается полная схема диссипативных эффектов в плазме. Например, показывается, что в случае компонент при различных температурах перенос импульса неизбежно связан с соответствующим переносом энергии. Затем показывается, как в предельном случае магнитной гидродинамики процесс электропроводности может быть включен в рамки диссипативных явлений.

(*) *Переведено редакцией.*