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Linearization Procedure for Cylindrically and Axially Symmetric Space-Times

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Abstract

An investigation of the vacuum Einstein gravitational field equations for cylindrically and axially symmetric space-times is presented which leads to an equivalent differential system involving a simple nonlinearity only. The case when this equivalent system is linear is analyzed in detail and two methods for generating solutions of the Einstein vacuum equations are set up. As a result, in the axially symmetric case the linearity of the equivalent system characterizes completely the Kramer–Neugebauer transforms of Papapetrou line elements. Accordingly, Weyl solutions are shown to generate exhaustively both Lewis and van Stockum solutions. Analogous results are obtained also in the cylindrically symmetric case.

§(1): *Introduction*

In recent years, much effort has been devoted to an attempt to develop methods for generating new solutions of the Einstein equations from the old ones; as a result, a great variety of such methods are now well established (see, e.g., [1]). In particular, cylindrically symmetric and axisymmetric stationary vacuum (or electrovacuum) fields have been analyzed extensively (see, e.g., [2] and references therein). Also, Harrison [3] and Neugebauer [4] derived a Bäcklund transformation which enabled them to generate a broad class of axisymmetric vacuum space-times, while Belinskii and Zakharov [5] used the inverse scattering problem technique to integrate the equations of gravity when the metric tensor depends on two coordinates only.

[1] The indices a, b take on the values 1 and 2, the signature of the metric is +2, and $c = 1$.

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In this paper I aim to examine the Einstein vacuum equations in the case of cylindrical and axial symmetry. It turns out that the gravitational equations are equivalent, in a natural way, to a differential system which, under suitable hypotheses, is linear. Hence, there exists a whole class of space-times, solutions to a genuinely nonlinear differential system, which enjoy the remarkable property that they may be calculated through a linear procedure. This feature makes possible the construction of two methods for generating solutions of the Einstein vacuum equations. As a result, in the case of axial symmetry, it is shown that the Weyl line elements [6] generate the whole class of Lewis [7] and van Stockum [8] solutions.

The paper is organized as follows. In Section 2, I review the Einstein gravitational equations and I set up an equivalent differential system involving a simple nonlinearity. The consequences of this system being linear are investigated in Section 3 and in Section 4 where diagonal and nondiagonal line elements are treated separately. In Section 5, I examine the results obtained so far by using canonical Weyl coordinates. It turns out that, for axially symmetric space-times, the equivalent system is linear if and only if its solutions are the Kramer–Neugebauer transforms [9] of Papapetrou solutions [10]. The cylindrically symmetric counterpart leads to a similar conclusion.

§(2): *A Differential System Equivalent to the Einstein Vacuum Equations*

Consider the metric tensor in the form[1]

$$ds^2 = f(z, t)\,(dz^2 - \lambda\, dt^2) + g_{ab}(z, t)\, dx^a\, dx^b \tag{1}$$

where $\lambda = \pm 1$. Henceforth, I denote the two-dimensional matrix (g_{ab}) by g and, without loss of generality, I choose the coordinate z so that [1]

$$\det g = \lambda z^2 \tag{2}$$

From the physical point of view, the metric (1) represents an axially symmetric stationary field when $\lambda = -1$ [11] and a cylindrically symmetric space-time when $\lambda = 1$ [1]. The complete system of the Einstein vacuum equations for the metric (1) splits into two sets of equations. The first set consists of a single matrix equation for the matrix g, namely,

$$(zg_{,z}g^{-1})_{,z} - \lambda(zg_{,t}g^{-1})_{,t} = 0 \tag{3}$$

a comma denoting partial differentiation. The second set expresses the function f by quadratures in terms of a given solution of (3) through the relations

$$(\ln f)_{,z} = -\frac{1}{z} + \frac{1}{4}\,\mathrm{Tr}\left(\frac{Z^2}{z} + \lambda z T^2\right) \tag{4}$$

$$(\ln f)_{,t} = \frac{1}{2}\,\mathrm{Tr}\,(ZT) \tag{5}$$

where

$$Z = zg_{,z}g^{-1} \tag{6}$$

$$T = g_{,t}g^{-1} \tag{7}$$

As remarked by Belinskii and Zakharov [5], owing to the identities

$$\operatorname{Tr} Z = z\,(\ln \det g)_{,z}, \qquad \operatorname{Tr} T = (\ln \det g)_{,t}$$

the trace of (3) shows that for every nonsingular solution g to (3) (for which, in principle, the condition (2) could not hold) the quantity $\det g$ satisfies the equation

$$[z\,(\ln \det g)_{,z}]_{,z} - \lambda[z\,(\ln \det g)_{,t}]_{,t} = 0$$

Using this result, it is easy to check the noteworthy property that the matrix

$$\bar{g} = z(\lambda \det g)^{-1/2} g \tag{8}$$

satisfies equation (3) as well as condition (2). Thus we are allowed not to worry about condition (2) during the calculations and to adopt the simple renormalization (8) of the final result so as to get the correct quantities.

My purpose is now to derive two first-order matrix equations defining the matrices Z and T. The first obvious equation follows from (3), (6), (7), and reads

$$Z_{,z} - \lambda z T_{,t} = 0 \tag{9}$$

The second one is the integrability condition for the relations (6), (7) with respect to g, that is,

$$Z_{,t} - zT_{,z} = [T, Z] \tag{10}$$

where $[T, Z] = TZ - ZT$ denotes the commutator between T and Z. It turns out that, once the matrices Z and T are solutions of the equations (9), (10), the matrix g, defined by the integrable system (6), (7), is solution of equation (3). In other words, the system (9), (10) is equivalent to equation (3) via the relations (6), (7).

Some troubles arise in connection with the symmetry of g because this property is not a consequence of the system (6), (7); to remedy this drawback additional requirements on the matrix g must be imposed. For the purpose of discussing this point, I let the matrices Z and T be solutions of equations (9), (10), and I seek solutions g of (6), (7) satisfying the further conditions

$$Tg = g\tilde{T} \tag{11}$$

$$Zg = g\tilde{Z} \tag{12}$$

a tilde denoting the transpose. The linear system (6), (7), (11), (12) enjoys the property that if g is a solution also $\tilde{g}$ is and so $g + \tilde{g}$ is a symmetric solution. In view of the assumptions on Z, T, the only nontrivial integrability conditions of

(6), (7), (11), (12) are

$$T_{,t}g - g\tilde{T}_{,t} = 0 \tag{13}$$

$$T_{,z}g - g\tilde{T}_{,z} = \frac{1}{z}[Z, T]g \tag{14}$$

Note that the analogous equations for the matrix Z are automatically verified, as a direct consequence of (13), (14) via (9), (10). As a final step, consider the enlarged system (6), (7), (11)-(14) for the matrix g. This linear system allows both g and $\tilde{g}$, and hence $g + \tilde{g}$, to be a solution, while its integrability conditions are identically verified. Therefore the result is that, once the matrices Z, T are solutions of (9), (10), it is always possible to find a symmetric solution of (6), (7) by imposing the conditions (11)-(14).

Although this procedure appears to be somewhat involved, almost all steps are linear. More precisely, whereas (3) is in general a complicated, essentially nonlinear, equation, the only nonlinearity present in the previous procedure is due to the term $[T, Z]$. This raises in a natural way the problem of analyzing what happens when $[T, Z] = 0$, namely, when equation (3) is equivalent to a linear system; this problem will be dealt with in the next sections.

Let me conclude this section with three remarks.

Remark 1. As pointed out in [5], the renormalization (8) of g implies a corresponding modification of Z and T, namely,

$$\overline{Z} = Z + z\{\ln[z(\lambda \det g)^{-1/2}]\}_{,z}I$$

$$\overline{T} = T + \{\ln[z(\lambda \det g)^{-1/2}]\}_{,t}I$$

where $\overline{Z}$ and $\overline{T}$ are defined in terms of $\overline{g}$ according to (6), (7) and I is the unit matrix. Then Z, T commute if and only if $\overline{Z}$, $\overline{T}$ do so. This confirms the correctness of accounting for (2) only at the end of the calculation via equation (8).

Remark 2. Assume $[T, Z] = 0$ and write the integrability conditions for the system (9), (10) with respect to the matrix Z. In so doing, the equation

$$(zT_{,z})_{,z} - \lambda(zT_{,t})_{,t} = 0 \tag{15}$$

for the matrix T is found which is the wave ($\lambda = 1$) or Laplace ($\lambda = -1$) equation in cylindrical symmetry. Analogously, looking at the integrability conditions of the system (9), (10) with respect to the matrix T yields

$$(z^{-1}Z_{,z})_{,z} - \lambda(z^{-1}Z_{,t})_{,t} = 0 \tag{16}$$

Remark 3. As noted above, the condition $[T, Z] = 0$ makes the system (9), (10) linear so its solutions may be linearly superposed. Now, in view of (6), (7), such a superposition results in a change of the matrix g which reflects, in a nonlinear way, the original superposition. Thus the problem at hand provides a further example of the so-called nonlinear superposition principle.

§(3): *The Diagonal Case*

I begin to investigate the consequences of the condition $[T, Z] = 0$ with letting Z and T be diagonal matrices. In this instance, account of (11), (12) leads to

$$T_{11} g_{12} = T_{22} g_{12}, \qquad T_{11} g_{21} = T_{22} g_{21}$$

$$Z_{11} g_{12} = Z_{22} g_{12}, \qquad Z_{11} g_{21} = Z_{22} g_{21}$$

whereby $g_{12} = g_{21} = 0$ provided either $Z_{11} \neq Z_{22}$ or $T_{11} \neq T_{22}$. In the remaining case when $Z_{11} = Z_{22}$, $T_{11} = T_{22}$, equations (6), (7) read

$$\gamma_{,z} = z^{-1} Z_{11} \gamma, \qquad \gamma_{,t} = T_{11} \gamma$$

γ being any of the entries of the matrix g. The general solution of these equations is

$$g = K \exp \left[\tfrac{1}{2} \left(\int z^{-1} Z_{11}\, dz + \int T_{11}\, dt \right] ,\right.$$

where K is a constant symmetric matrix. Thus there exists a constant matrix P, with $P\tilde{P} = I$, such that $\tilde{P} g P$ is diagonal; hence the coordinate transformation

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \longrightarrow P \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

makes the matrix g into its diagonal form. In conclusion I have proved that no generality is lost by allowing the matrix g to be diagonal.

Owing to the diagonal form of g, equations of Section 2 take on simpler expressions. Indeed by putting

$$g = \begin{pmatrix} z^2 \exp(-\psi) & 0 \\ 0 & \lambda \exp \psi \end{pmatrix} \tag{17}$$

equation (3) becomes

$$(z\psi_{,z})_{,z} - \lambda z \psi_{,tt} = 0 \tag{18}$$

while equations (6), (7) are equivalent to two scalar relations defining two functions, ζ and τ, given by

$$\zeta = z\psi_{,z} \tag{19}$$

$$\tau = \psi_{,t} \tag{20}$$

Accordingly, equations (9), (10) are to be viewed as equations for ζ and τ. Remarkably, comparison between (15) and (18) shows that ψ and τ satisfy the same equation; this suggests the possibility of associating a new solution with any solution of (18). Precisely, let ψ^* be a given solution of (18). The method consists in setting

$$\tau = \psi^*$$

and in solving equations (9), (10) to find the function ζ. Then, the solution sought ψ follows by integrating the system (19), (20).

As an example, consider the trivial flat space-time solution of (18), that is,

$$\psi = 0$$

Following the procedure indicated above, I find that the solution associated with $\psi = 0$ is

$$\psi = k \ln z \tag{21}$$

k being a real constant. This is the Kasner-Levi-Civita solution [12, 13] in a nonsynchronous coordinate system. This example shows that the Kasner solution is in fact insensitive to the value of λ, as previously remarked by Geroch [14]. Moreover, in view of the linearity of (18), the solution (21) will always appear when searching for new solutions in that ζ is determined up to a constant. This may be phrased by stating that solutions of the Einstein vacuum equations for the diagonal metric tensor (1) are interpretable as "modulated" Kasner space-times (in this connection see [15]). In view of this, appeal to hereditary properties of limits of space-times [16, 17] shows that solutions of (18), (4), (5) are at most of type D if $k = 1$, of type I for every $k \neq 0, 1, 2$, whereas this method does not provide information in the flat Kasner case, namely, when $k = 0$ or $k = 2$.

§(4): *The Nondiagonal Case*

In this section I shall examine the case when the matrices Z and T commute without being diagonal. To this end I observe that the matrices

$$Z = \begin{pmatrix} p + \alpha r & \beta r \\ \gamma r & p \end{pmatrix}, \quad T = \begin{pmatrix} q + \alpha s & \beta s \\ \gamma s & q \end{pmatrix} \tag{22}$$

where $p, q, r, s, \alpha, \beta, \gamma$ are arbitrary quantities, are the most general 2×2 commuting matrices. Thus the problem at hand consists now in solving equations (9), (10) when Z and T are given the forms (22). However, although equations (9), (10) are linear, it seems that $p, q, r, s, \alpha, \beta, \gamma$ are not in fact solutions to a linear system. The following argument will convince us that this is not so.

Write first the matrix system (9), (10) in its scalar form

$$\mu_{,z} - \lambda z \nu_{,t} = 0 \tag{23}$$

$$\mu_{,t} - z \nu_{,z} = 0 \tag{24}$$

where μ is any of the entries of Z and ν is the corresponding entry of T. As a direct consequence of (22), the pair $\mu = p$, $\nu = q$ satisfies equations (23), (24).

Moreover, it is easy to show that the pair $\mu = r$, $\nu = s$ satisfies equations (23), (24) if and only if α, β, γ are constants. Indeed, if r, s are solutions to the system (23), (24), taking account of (22) implies that α, say, is solution to the system

$$r\alpha_{,z} - \lambda zs\alpha_{,t} = 0$$

$$zs\alpha_{,z} - r\alpha_{,t} = 0$$

which is equivalent to

$$\alpha_{,z} = 0, \qquad \alpha_{,t} = 0, \qquad \text{or} \quad \alpha = \text{const}$$

in that the determinant $-r^2 + \lambda s^2 z^2$ is always different from zero in view of the assumptions on r and s. As identical results hold for β and γ, we lose no generality by assuming α, β, γ to be constants.

In conclusion, the original problem reduces to solving the linear scalar system (23), (24) for the pairs (p, q), (r, s) and to constructing the matrices Z and T according to the rule (22). Thus, having at disposal the matrices Z, T, solutions to (9), (10), it follows from Section 2 that the required symmetric nonsingular matrix g may be calculated by a direct integration of the linear system (6), (7), (11)-(14). It is worthy of note that I have found here a linear counterpart of the genuine nonlinear equation (3).

The previous results indicate a method of generating a nondiagonal solution from any diagonal one. Precisely, let g^0, Z^0, T^0 be a known diagonal solution of the system (6), (7), (9)-(14). The method relies on seeking a nondiagonal solution Z, T whose diagonal entries coincide with those of Z^0, T^0, respectively. Two cases arise. First, if $Z^0_{11} = Z^0_{22}$ and $T^0_{11} = T^0_{22}$, it follows from (22) that $\alpha = 0$. So the desired result is

$$Z = \begin{pmatrix} Z^0_{11} & \beta r \\ \gamma r & Z^0_{22} \end{pmatrix}, \qquad T = \begin{pmatrix} T^0_{11} & \beta s \\ \gamma s & T^0_{22} \end{pmatrix} \tag{25}$$

where the pair (r, s) is an arbitrary solution of the system (23), (24) and β, γ are constants. Second, when $Z^0_{11} \neq Z^0_{22}$ or $T^0_{11} \neq T^0_{22}$, account of (22) shows that $\alpha \neq 0$ and therefore no loss of generality by setting $\alpha = 1$. In this instance equations (22) yield

$$r = Z^0_{11} - Z^0_{22}, \qquad s = T^0_{11} - T^0_{22} \tag{26}$$

which, in view of the linearity of the system (9), (10) satisfy equations (23), (24); this in turn implies that β, γ are constants. The solution takes again the form (25), r and s being given by (26). The final step is to find g when Z and T are given by (25) and possibly (26).

Although more trivial, it should be noted that a diagonal solution may be associated with any nondiagonal one. Indeed, as a direct consequence of the linearity of the system (23) and in view of (22), given a nondiagonal solution Z, T, the diagonal counterpart is obtained simply by dropping out the nondiagonal entries of Z and T, which is tantamount to putting $\beta = \gamma = 0$. This feature ultimately tells us that, when Z and T commute, there exists a one-to-one correspondence between diagonal and nondiagonal line elements.

§(5): *Comparison with the Current Approach*

So as to arrive at a deeper understanding of the condition $[T, Z] = 0$, it is convenient to refer the metric tensor to the Weyl canonical coordinates (see, e.g., [1] p. 195). In these coordinates the line element (1) reads

$$ds^2 = \exp(-2U + 2k)(dz^2 - \lambda\, dt^2) + z^2 \exp(-2U)(dx^1)^2 + \lambda \exp(2U)(dx^2 + A\, dx^1)^2 \quad (27)$$

whereby the explicit form of the matrices Z and T can be calculated straightforwardly. Precisely, defining the new function

$$S = -U + \tfrac{1}{2} \ln z \quad (28)$$

I obtain from (6), (7) the following expressions for Z and T:

$$\begin{aligned} Z_{11} &= z[z^{-1} + 2S_{,z} + \lambda A A_{,z} \exp(-4S)] \\ Z_{12} &= z[-4AS_{,z} + A_{,z} - \lambda A^2 A_{,z} \exp(-4S)] \\ Z_{21} &= z\lambda A_{,z} \exp(-4S) \\ Z_{22} &= z[z^{-1} - 2S_{,z} - \lambda A A_{,z} \exp(-4S)] \end{aligned} \quad (29)$$

$$\begin{aligned} T_{11} &= 2S_{,t} + \lambda A A_{,t} \exp(-4S) \\ T_{12} &= -4AS_{,t} + A_{,t} - \lambda A^2 A_{,t} \exp(-4S) \\ T_{21} &= \lambda A_{,t} \exp(-4S) \\ T_{22} &= -2S_{,t} - \lambda A A_{,t} \exp(-4S) \end{aligned} \quad (30)$$

I am now in a position to draw the consequences of the assumption $[T, Z] = 0$. As a direct use of (29), (30) shows, the matrices Z and T commute if and only if the condition

$$S_{,z} A_{,t} = S_{,t} A_{,z} \quad (31)$$

holds. Hence, since both S and A depend on z, t only, it turns out that S and A must be functionally dependent, that is to say,

$$S = S(A) \quad (32)$$

On account of (32), from the (21) and (11)-(22) components of the field equations (9) (cf. also [1], pp. 198 and 223) one learns that

$$\exp(4S) = -\lambda A^2 + C_1 A + C_2 \quad (33)$$

while it follows from (29), (30), (33) that the constants C_1, C_2 appearing in (33) satisfy the equations

$$Z_{11} - Z_{22} = \lambda C_1 Z_{21}, \quad Z_{12} = \lambda C_2 Z_{21} \tag{34}$$

$$T_{11} - T_{22} = \lambda C_1 T_{21}, \quad T_{12} = \lambda C_2 T_{21} \tag{35}$$

As a result, the condition (33) is completely equivalent to $[T, Z] = 0$. This, in turn, implies that, in the instance $\lambda = -1$, all solutions to the linear problem (6), (7), (9)-(14) are the Kramer-Neugebauer transforms [9] of Papapetrou solutions [10], whereas the case $\lambda = 1$ constitutes the cylindrically symmetric counterpart.

Finally, it is worth commenting briefly on the method, described at the end of the previous section, which allows nondiagonal and diagonal solutions to be related to each other. In essence, a straightforward inspection of (29), (30) leads to the following conclusions. First, let $Z^0_{11} = Z^0_{22}$, $T^0_{11} = T^0_{22}$ (and hence, necessarily, $Z_{11} = 1$, $T_{11} = 0$, which correspond to the Kasner type-D space-time); then, in view of the arbitrariness of r and s, for every ψ solution to equation (18), the relation

$$\int \exp(-4S)\, dA = \lambda\psi \tag{36}$$

provides the function $A = A(z, t)$. Of course, owing to equation (33), the left-hand side of (36) is a known function of A. Second, let $Z^0_{11} \neq Z^0_{22}$ or $T^0_{11} \neq T^0_{22}$; in this case, representing the diagonal solution by means of equation (17), it follows that the function A is given again by (36), where ψ is now fixed through the condition (17).

A noteworthy consequence of this analysis is that diagonal solutions generate the whole class of metric tensors for which $[T, Z] = 0$. In particular, introducing the quantity $h = 4C_2 - C_1^2$, in the case $\lambda = -1$ such a class collects together Weyl ($h < 0$), Lewis ($h > 0$), and van Stockum ($h = 0$) solutions (cf. [1], p. 204). Therefore the conclusion is that Weyl (diagonal) solutions generate both Lewis and van Stockum solutions, thereby generalizing a previous result by Tanabe [18]. An analogous conlcusion is arrived at when $\lambda = 1$.

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References

1. Kramer, D., Stephani, H., McCallum, M., and Herlt, E. (1980). *Exact Solutions of Einstein Field Equations*, VEB Deutscher Verlag der Wissenschaften, Berlin.

2. Kinnersley, W. (1980). *J. Math. Phys.*, **21**, 2231.
3. Harrison, B. K. (1978). *Phys. Rev. Lett.*, **41**, 1197.
4. Neugebauer, G. (1979). *J. Phys. A*, **12**, L67.
5. Belinskii, V. A., and Zakharov, V. E. (1979). *Sov. Phys. JETP*, **48**, 985.
6. Weyl, H. (1917). *Ann. Phys. (Leipzig)*, **54**, 117.
7. Lewis, T. (1932). *Proc. R. Soc. London Ser. A*, **136**, 176.
8. van Stockum, W. J. (1937). *Proc. R. Soc. Edinburgh A*, **57**, 135.
9. Kramer, D., and Neugebauer, G. (1968). *Commun. Math. Phys.*, **10**, 132.
10. Papapetrou, A. (1953). *Ann. Phys. (Leipzig)*, **12**, 309.
11. Reina, C., and Treves, A. (1976). *Gen. Rel. Grav.*, **7**, 817.
12. Kasner, E. (1921). *Am. J. Math.*, **43**, 217.
13. Levi-Civita, T. (1919). *Rend. Accad. Lincei*, **28**, 101.
14. Geroch, R. (1967). *J. Math. Phys.*, **9**, 450.
15. Bampi, F. (1981). *Boll. Un. Mat. Ital.*, **18 A**, L83.
16. Geroch, R. (1969). *Commun. Math. Phys.*, **13**, 180.
17. Bampi, F., and Cianci, R. (1980). *Int. J. Theor. Phys.*, **19**, 211.
18. Tanabe, Y. (1979). *J. Math. Phys.*, **20**, 1486.