RELATIVISTIC HEAT EQUATION IN CAUSAL NONSTATIONARY THERMODYNAMICS

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A hidden variable approach to nonstationary relativistic thermodynamics is developed thoroughly in the case of heat conducting fluids. The results show that the theory is consistent with and more general than other ones that appeared recently in the literature.

In recent years much attention has been devoted to topics concerning nonstationary relativistic thermodynamics especially in connection with astrophysical and cosmological problems. More specifically, the attention has been focused on dissipative processes so as to explain the high regularity in the structure of the Universe at large scale (cf. ref. [1] and references cited therein). In this context, much research has been undertaken in an attempt to produce a proposal overcoming the drawbacks of the Navier–Stokes–Fourier theory whereby the signals are propagated at infinite speed. Such is the case, for example, of the nonstationary irreversible thermodynamics elaborated by Israel and Stewart [2,3]; as shown in ref. [4], the adoption of this causal thermodynamics leads to crucial consequences on the evolution of model universes. While particularly awkward in a relativistic theory, infinite propagation speed is annoying at the classical level as well; that is why classical irreversible thermodynamics has been investigated by many workers (cf. e.g. refs. [5,6]).

A causal nonstationary thermodynamics has been set up by ourselves via a hidden variable approach both in the classical [7,8] and in the relativistic [9,10] framework. Our aim here is twofold: to give further insight into the thermodynamics with hidden variables and to make a timely and helpful comparison with the extended irreversible thermodynamics by Lebon et al. [6] through the corresponding causal evolution for heat conduction [11].

Henceforth, the indices $\rho, e, \theta, q, \Lambda$ denote partial derivatives. A superposed dot stands for the covariant derivative along the fluid flow, i.e. $\dot{q}_\mu = u^\nu q_{\mu;\nu}$, while $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the spatial projector.

Briefly, Pavon et al. [11] allow the specific entropy function $\hat{s}$ of a fluid to depend on the heat flux q^μ, besides on the mass density ρ and the internal specific energy e. Then, upon setting

$$\hat{s}_{q_\mu} = \alpha(e, \rho)(\rho T)^{-1} q^\mu, \tag{1}$$

the *temperature T being defined* by

$$T^{-1} = \hat{s}_e, \tag{2}$$

they exploit the corresponding entropy balance equation and, as the most simplifying assumption, they find that

$$q^\mu = -\kappa h^{\mu\nu}(\lambda_\nu - \alpha T \dot{q}_\nu), \tag{3}$$

where κ is the thermal conductivity and λ_ν is the relativistic temperature gradient $T_{,\nu} + T\dot{u}_\nu$. The requirement that eq. (3) reduces to the well-known Maxwell–Cattaneo equation in the comoving frame amounts to setting

$$\alpha = -\tau(\kappa T)^{-1}, \tag{4}$$

τ denoting the proper relaxation time of the process. This is sufficient for the comparison we have in mind.

Generalizing the customary outlook, we say that a material with hidden variables consists of a set of response functions

$$\phi = \phi(y, \Lambda)$$

and of a function f governing the growth of the hidden variables, namely

$$\dot{\Lambda} = f(y, z, \Lambda, \Lambda_{;\mu}).$$

Essentially the distinction between the physical variables y and z is that the latter vanish at equilibrium. Such a distinction is substantiated by the fact that the explicit dependence of ϕ on z leads unavoidably to the paradox of infinite propagation speed. In addition to other results, this scheme provides Müller–Israel's theory [12,2] as a particular case [10].

When dealing with non-viscous heat conducting fluids, y may be identified with the pair (θ, ρ), θ being the absolute temperature, and z with λ. For our purposes it is enough that we choose Λ as a spatial vector, $\Lambda^\mu u_\mu = 0$, and f as a linear function, that is to say

$$h_\mu{}^\nu \dot{\Lambda}_\nu = a\lambda_\mu + b\Lambda_\mu;$$

here the coefficients a, b are assumed to be independent of θ, ρ. The requirement that a pair (λ^0, Λ^0) be asymptotically stable at fixed λ^0 forces b to be negative; setting $b = -\tau$ the parameter τ takes on the meaning of relaxation time. So, upon renormalizing the hidden variable, namely $(a\tau)^{-1}\Lambda \to \Lambda$, we have

$$h_\mu{}^\nu \dot{\Lambda}_\nu = \tau^{-1}(\lambda_\mu - \Lambda_\mu). \tag{5}$$

The response ϕ – which may be identified with the set (ψ, s, p, q), ψ and p being the free energy and the pressure – is restricted by the Clausius–Duhem inequality whereby

$$-\rho(\dot{\psi} + s\dot{\theta}) - pu^\mu{}_{;\mu} - \theta^{-1} q^\mu \lambda_\mu \geqslant 0$$

must hold identically. As $\psi = \psi(\theta, \rho, \Lambda)$ and $\dot{\rho} = -\rho u^\mu{}_{;\mu}$, substitution of eq. (5) yields

$$-\rho(\psi_\theta + s)\dot{\theta} + (\rho^2 \psi_\rho - p)u^\mu{}_{;\mu} - (\theta^{-1} q^\mu + \rho\tau^{-1}\psi_{\Lambda_\mu})\lambda_\mu + \rho\tau^{-1}\psi_{\Lambda_\mu}\Lambda_\mu \geqslant 0. \tag{6}$$

The independence of the hidden variable Λ of the present values of the physical variables $\theta, \dot{\theta}, \rho, u^\mu{}_{;\mu}, \lambda$ [7,9] allows us to say that (6) holds only if

$$s = -\psi_\theta, \quad p = \rho^2 \psi_\rho, \quad q^\mu = -\rho\theta\tau^{-1}\psi_{\Lambda_\mu}, \tag{7}$$

$$\psi_{\Lambda_\mu}\Lambda_\mu \geqslant 0. \tag{8}$$

Whenever the function ψ satisfies the inequality (8), the response functions s, p, q, as defined by eqs. (7), are automatically consistent with the second law of thermodynamics. Now, on the basis of physical arguments, we choose a particular function ψ among the admissible ones. Specifically, set

$$\psi(\theta, \rho, \Lambda) = \Psi(\theta, \rho) + \kappa\tau(2\theta\rho)^{-1}\Lambda^\mu\Lambda_\mu. \tag{9}$$

Hence eqs. (7) become

$$s = -\Psi_\theta + \kappa\tau(2\theta^2\rho)^{-1}\Lambda^\mu\Lambda_\mu, \quad q^\mu = -\kappa\Lambda^\mu, \tag{10}$$

while the inequality (8) gives $\kappa \geqslant 0$. Note that in the case when the temperature gradient is held constant eq. (5) asymptotically yields $\Lambda = \lambda$ and, what is more, eq. (10) becomes Fourier's law of heat conduction. Finally, the definition $e = \psi + \theta s$ provides

$$e = \Psi - \theta\Psi_\theta + \kappa\tau(\rho\theta)^{-1}\Lambda^\mu\Lambda_\mu. \tag{11}$$

The comparison with the relations (1)–(4) is not immediate because the energy e depends on the hidden variable Λ besides on the equilibrium quantities θ, ρ. As a first step, observe that use of eqs. (10) and (11) leads to

$$\hat{s}_e = s_\theta / e_\theta = \theta^{-1}$$

and hence $T = \theta$. Then, in view of the identity

$$\hat{s}_{q_\mu} = s_{q_\mu} - s_\theta e_{q_\mu}/e_\theta,$$

a direct calculation provides

$$\hat{s}_{q_\mu} = -\tau(\theta^2\rho\kappa)^{-1} q^\mu.$$

On account of eq. (4), this result coincides exactly with expression (1) thus showing how the flexibility of the hidden variable approach permits us to obtain the conclusion of ref. [11] in the important special case (9).

The fact that the two theories are so strictly related is satisfactory from a general viewpoint but even more in conjunction with the structure of the entropy function (10). Indeed, as shown above, the entropy is unavoidably dependent on the hidden variable Λ and the corresponding term plays a vital role in our deduction. The importance of this aspect is strengthened by the property of the Clausius–Duhem inequality whereby, even accounting for further irreversible phenomena, the entropy is affected by hidden variables only through the one describing heat conduction [7,10].

Beyond the consistency of the different approaches to nonstationary thermodynamics (cf. also ref. [10]) to our mind one element weighs in favour of the hidden variables. Such is the case of wave propagation problems where the structure of the evolution equations allows us to regard the hidden variables as continuous quantities across the discontinuity fronts [9].

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