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Objectivity and Objective Time Derivatives in Continuum Physics[1]

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The role played by objectivity in continuum physics is reexamined in an attempt to establish fully its deep connection with classical and relativistic time derivatives. The way of distinguishing one element in the class of objective time derivatives may depend on the particular problem of interest; this is emphasized in conjunction with material relaxation phenomena described via hidden variable evolution equations.

1. INTRODUCTION

The principle of objectivity has drawn the attention of many researchers in the last decade. Basically, the growing interest in this topic may be explained by the following observations. As the literature concerning continuum mechanics shows, severe restrictions are placed by objectivity on the form of constitutive equations (see, e.g., Ref. 1, Part A, Ref. 2, Part II, Chapter I). Even in view of this fact, extreme care must be exercized in the elevation of an accepted rule to the peerage of a fundamental principle.[3] Moreover, while in classical mechanics a unique statement is agreed upon, as yet there is no such unique formulation in relativistic physics.

One of the points to emerge from this paper is the deep connection between the principle of objectivity and objective time derivatives. In synthesis, this connection may be outlined as follows. First, in classical mechanics the mathematical formulation of the objectivity principle yields unambiguously the class of objective time derivatives. Furthermore, in relativity the interrelation is made evident by the fact that proper statements

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of the principle distinguish relevant objective time derivatives, and vice versa.

In addition to being an interesting topic on its own, the problem of selecting appropriate time derivatives in continuum physics gets its practical motivation from the need to describe the macroscopic behavior of materials via evolution equations for hidden (or internal) variables. Such is the case, for example, of reacting mixtures where each hidden variable represents the degree of advancement of a chemical reaction.[4] Generally, hidden variables may be viewed as averaged global parameters which, in an average approximate manner, account for macroscopic phenomena resulting from microscopic processes. This idea is made clear by two works of Kluitenberg[5] where the Debye theory for dielectric relaxation phenomena in polar fluids and Snoeck's relaxation equation in magnets are achieved through the hidden variable approach. Magnetic relaxation phenomena in finitely deformable ferromagnets are investigated by Maugin,[6] who selects as vectorial hidden variable the intrinsic spin of quantum mechanical origin. Hidden variables have also been used successfully for describing involved mechanical behaviors of continua such as elastoviscoplastic bodies.[7] Finally, we mention that recent applications[8–10] have revealed the hidden variables to be suitable for representing viscosity and heat conduction without suffering from the unpleasant paradox of infinite speed of propagation arising from the absence of relaxation phenomena in the standard theory.

The plan of the paper is as follows. We begin by surveying the main features of objectivity (Section 2). Then, in Section 3, we set up a self-contained approach to classical time derivatives. In so doing we emphasize the relationship between the invariance under the Euclidean group and the mathematical structure of objective time derivatives. Following along analogous lines, in Section 4, we characterize the class of relativistic time derivatives, which is to be viewed as the direct counterpart of the classical one. In Section 5 we point out how purely theoretical arguments cannot distinguish any objective time derivative from the others. Accordingly, we advocate recourse to experimental checks such as those involving wave propagation in materials with hidden variables. Finally, some open controversial points are discussed in Section 6.

2. OBJECTIVITY IN CONTINUUM PHYSICS

It is customarily accepted that a constitutive theory is admissible provided it complies with the content of a suitable set of principles (see, e.g., Ref. 2, p. 134; Ref. 11, §293; Ref. 12, §4). Among them, the principle of objectivity (or material frame indifference) has been recently the subject of several papers[3,13–23] which show how its precise content and range of validity are still a controversial matter. Because of such a controversy, it seems worth summarizing here the main themes pertaining to the topic of objectivity.

Look first at the classical framework. In this regard a short but comprehensive history of this principle is provided by Truesdell and Noll (Ref. 1, §19). Presently, according to most works in continuum mechanics, the principle of objectivity is stated so that constitutive equations must be form-invariant under the Euclidean group. This group is a nonfinite group consisting of all transformations of the time t and of the Cartesian coordinates x^i in the current configuration such that[3]

$$\begin{aligned} x'^i &= Q^i{}_j(t)\, x^j + c^i(t) \\ t' &= t + a \end{aligned} \tag{1}$$

where a is a constant, (c^1, c^2, c^3) is an arbitrary point function, and $Q^i{}_j$ is an arbitrary orthogonal matrix function of the time t. To make the meaning of the objectivity statement clearer, designate by objective tensors of rank n quantities $A^{i_1 \cdots i_n}$ which transform according to the tensor rule

$$A'^{i_1 \cdots i_n} = Q^{i_1}{}_{j_1} \cdots Q^{i_n}{}_{j_n} A^{j_1 \cdots j_n} \tag{2}$$

under the action of (1). So the principle of objectivity demands that constitutive equations involve objective tensors only. Needless to say, because of the arbitrary time dependence of $c^i(t)$ and $Q^i{}_j(t)$, many tensors in the ordinary sense are not objective tensors. For example, the velocity vector $v^i = \dot{x}^i$ and its spatial gradient $L_{ij} = v_{i,j}$ are not objective; this point will be returned to in Section 3.

Before going further some remarks are in order. Strictly speaking, the principle of objectivity involves form invariance under general transformations between rigid frames, namely both under the Euclidean group (1) and under the arbitrary time-independent change of coordinates within any rigid frame. In fact, time-independent changes of coordinates do not introduce any difficulties into the problem at hand; hence we investigate objectivity topics in conjunction with the Euclidean group through Cartesian coordinates only. Yet, in order to account naturally for different behaviors of covariant and controvariant components, we distinguish lower and upper indices.

According to the recent literature on the subject, it seems that the main problem is to test the plausibility of the objectivity principle by examining the behavior of material functions such as stress and heat flux. For instance, with the purpose of exhibiting real material properties in contrast with

[3] Latin indices run from 1 to 3, Greek indices run from 0 to 3, a comma denotes a partial derivative, a semicolon denotes a covariant derivative. A superposed dot stands for the material time derivative.

objectivity, Edelen and McLennan[3] considered Burnett's equations for the stress tensor and the heat flux. Then they emphasized that, owing to the presence of the spin tensor[4] $W_{ij} = L_{[ij]}$, the contradiction sought is evident. Müller[13] and Söderholm[14] as well looked for realistic constitutive equations through kinetic approaches. They both found that, in the case of a steady rotation of a gas of Maxwellian molecules at substantially constant temperature, $\dot{\theta} = 0$, the heat flux q^i takes the form

$$q^i = -\kappa(\delta^{ij} + \tau W^{ij})\theta_{,j}$$

whereby the heat conductivity would be a tensor with a frame-dependent skew-symmetric part. Similar effects were obtained by Atkin and Fox[24] in connection with polar fluids.

A counterargument against these conclusions has been given by Wang,[15] who pointed out that kinetic theory cannot provide conclusive statements upon the validity of the objectivity principle. Two reasons follow. First, there is no proof that the formal expressions of kinetic theory converge so as to justify the leading terms as adequate approximations. Second, there is no proof that the limits of the expansions have the same qualitative properties as the leading terms, especially with regard to objectivity. But, apart from these matters of rigor, as outlined by Wang and investigated in detail by Truesdell, the contradictions appearing in Refs. 3, 13, and 14 arise because these works do not account completely for Maxwell's consistency theorem,[25] which asserts that "the gross fields associated with any solution of the Maxwell–Boltzmann equation satisfy the condition of gross balance of mass, momentum, and energy." Indeed, the analysis of the restrictions placed by Maxwell's consistency theorem[16] shows that, within the scheme adopted in Refs. 3, 13, and 14, the pressure deviator (viscous stress) and the heat flux must vanish, thus eliminating the aforementioned contradictions. It seems then that no argument allows us to cast doubts upon the objective character of the material functions.

This assertion is substantiated also by the literature concerning relativistic continuum physics in that the objective character of material functions is usually assumed. However, relativistic researchers are still debating about what the objectivity really is. No one should be surprised by this circumstance; the difficulty is due to the lack of a sound generalization of the Euclidean group, or the idea underlying it because of the genuine four-dimensional structure of relativity. Attempts to obtain a principle of objectivity within special relativity have been carried out by Grot and Eringen, Bragg, and

[4] As usual, round brackets denote symmetrization, while square brackets denote skew symmetrization.

Söderholm. In synthesis, Grot and Eringen[26] postulated that constitutive equations must be covariant under the Lorentz group. Following different viewpoints, Bragg[19] and Söderholm[21] introduced definitions of equivalent motions and then defined a constitutive equation as objective if the corresponding value for all equivalent motions is the same. Unfortunately, Grot and Eringen's principle does not reduce to the classical one, while to date not many consequences of Bragg's and Söderholm's theories have been elaborated.

Valuable endeavors to achieve an objectivity principle in general relativity have been performed by Bressan, Maugin, Oldroyd, and Lianis. In 1964 Bressan[17,18] proposed that constitutive equations must involve absolute quantities. In other words, he advocated the exclusive use of quantities relative to a reference state, thus extending to relativity Pipkin and Rivlin's[27] and Rivlin's[28] classical statements. By close analogy with the classical scheme, Maugin[29] asserted that "constitutive equations must be invariant with respect to superposition of an arbitrary local Herglotz–Born rigid body motion." According to Maugin himself, the motivation for such a principle is delivered by the results of its application to constitutive functionals. Oldroyd and Lianis faced the question at a more general level. Basically, they established an epistemological principle whereby a constitutive equation describing physical conditions at a neighborhood of a material particle contains only information which is irrelevant to the particle motion, relative to any observer, and to the particle position in spacetime and which can be obtained by physically acceptable measuring operations. It is an immediate mathematical consequence of this principle that constitutive equations involve space tensors only (see, e.g., Ref. 26) and that material functions are objective. Bearing this in mind, Oldroyd[20] made the principle operative by distinguishing a (convected) coordinate system, and by relating objective quantities to such a system (rheological invariance), while Lianis[22] fixed his attention on the associated Fermi frame subsequently reduced to the proper rigid frame.[23] Accordingly, Bressan's standpoint, too, may be viewed within this broader context.

We adhere to the epistemological principle expressed above. So our problem is now to decide about the way of realizing it. Basically, we must single out a triad of spatial vectors to which refer the material functions. In other words, we must fix the evolution of the triad along the streamlines of the body in a manner relevant to accounting for the constitutive properties of the material at hand. This point is equivalent to choosing a transport law along the flow of the body and hence to selecting a distinguished time derivative. That is why the problem of objectivity will be returned to shortly after a preliminary analysis of the derivatives both in the classical and in the relativistic frameworks.

3. CLASSICAL OBJECTIVE TIME DERIVATIVES

The topic we are going to investigate has aspects which make it interesting. First, the material time derivative, when viewed as a rule assigning to every frame the time derivative relative to that frame, is nonobjective. Nevertheless, as we shall see in a moment, the class of objective time derivatives is very large indeed. Second, objective tensors may be derived from nonobjective ones too. Such is the case of the symmetric part of the velocity gradient $D_{ij} = L_{(ij)}$. For, according to (1) we find that

$$v'^i = Q^i{}_r v^r + \dot{Q}^i{}_r x^r + \dot{c}^i \tag{3}$$

Hence $L'_{ij} = \partial v'_i / \partial x'^j$ is given by

$$L'_{ij} = Q_i{}^r Q_j{}^s L_{rs} + \dot{Q}_{ir} Q_j{}^r \tag{4}$$

The appearance of $\dot{Q}^i{}_r x^r + \dot{c}^i$ and $\dot{Q}_{ir} Q_j{}^r$ makes v^i and L_{ij} nonobjective tensors in that they do not satisfy (2). However, in view of the skew symmetry of $\dot{Q}_{ir} Q_j{}^r$, we get the desired conclusion

$$D'_{ij} = Q_i{}^r Q_j{}^s D_{rs} \tag{5}$$

together with the transformation law

$$W'_{ij} = Q_i{}^r Q_j{}^s W_{rs} + \dot{Q}_{ir} Q_j{}^r \tag{6}$$

for the spin tensor W_{ij}.

Now, to examine the topic of objective time derivatives in a systematic way, it is convenient to premise some general concepts.

A derivation[30] ∂ of a tensor algebra $\mathscr{D}$ (considered as an algebra over $\mathbb{R}$) is a rule $\partial\colon \mathscr{D} \to \mathscr{D}$ such that, for every $Y, Z \in \mathscr{D}$, $\alpha, \beta \in \mathbb{R}$,

$$\partial(\alpha Y + \beta Z) = \alpha\, \partial(Y) + \beta\, \partial(Z) \qquad \text{linearity}$$

$$\partial(Y \otimes Z) = \partial(Y) \otimes Z + Y \otimes \partial(Z) \qquad \text{Leibniz' rule}$$

If, further, $\mathscr{D}$ possesses a contraction C, then ∂ enjoys the condition

$$\partial(C(Y, Z)) = C(\partial(Y), Z) + C(Y, \partial(Z)) \quad \text{commutativity with contractions}$$

On the other hand, a linear operator ∂, which acts on scalars and (either controvariant or covariant) vectors according to Leibniz' rule, may be extended uniquely to a derivation of the entire tensor algebra $\mathscr{D}$, commuting with contractions and preserving type of tensors. Then, to save writing, henceforth our attention will be confined mainly to scalars and vectors.

Let $\mathscr{O}$ be the algebra of objective (Euclidean) tensors; any derivation of $\mathscr{O}$ with respect to the time is termed an objective time derivative and is denoted by a superposed spot. Thus, under the transformation (1),

$$\mathring{A}'^{i_1\cdots i_n} = Q^{i_1}{}_{j_1} \cdots Q^{i_n}{}_{j_n} \mathring{A}^{j_1\cdots j_n}$$

for every tensor $A^{i_1\cdots i_n} \in \mathscr{O}$. To be operative, we have to fix unambiguously the time dependence of objective tensors. Look at a body $\mathscr{B}$ whose particles occupy the region $\mathscr{R}$ in a suitable reference configuration. Let $x^i = x^i(X^K, t)$ denote the position at the time t, with respect to a fixed Euclidean frame of reference $\mathscr{F}$, of the particle labeled by $\mathbf{X} = (X^1, X^2, X^3) \in \mathscr{R}$. On account of the law $x^i = x^i(\mathbf{X}, t)$, every Euclidean tensor $A^{i_1\cdots i_n}$ turns out to be expressed by a set of functions of the form $A^{i_1\cdots i_n} = A^{i_1\cdots i_n}(\mathbf{X}, t)$ relative to $\mathscr{F}$. The derivative with respect to t, the particle $\mathbf{X}$ being held constant, is just the material time derivative. The first step toward a characterization of a general objective derivative is to define $\mathring{f}$ for any Euclidean scalar f. By analogy with the current literature we set[5]

$$\mathring{f} = \dot{f} \tag{7}$$

As to the spot derivative of a vector A^i, the most general definition compatible with (7) is

$$\mathring{A}^i = \dot{A}^i + K^i{}_j A^j \tag{8}$$

with respect to every frame $\mathscr{F}$. So, for a mixed tensor $A^{i\cdots}{}_{j\cdots}$ we find that

$$\mathring{A}^{i\cdots}{}_{j\cdots} = \dot{A}^{i\cdots}{}_{j\cdots} + K^i{}_p A^{p\cdots}{}_{j\cdots} + \cdots - K^p{}_j A^{i\cdots}{}_{p\cdots} - \cdots \tag{9}$$

which justifies the distinction between covariant and contravariant indices and emphasizes that the spot derivative in general does not commute with the metric. The time-dependent matrix $K^i{}_j$ appearing in (8) may be split into its objective and nonobjective parts $S^i{}_j$ and $H^i{}_j$ as

$$K^i{}_j = S^i{}_j + H^i{}_j \tag{10}$$

It is the transformation law of the material time derivative which lends importance to this splitting. To make this point clear, consider (2) in the case of a vector and differentiate with respect to t with $\mathbf{X}$ held constant. We get the relation

$$\dot{A}'^i = Q^i{}_j \dot{A}^j + \dot{Q}^i{}_j A^j = Q^i{}_j \dot{A}^j + \dot{Q}^i{}_p Q_j{}^p A'^j \tag{11}$$

[5] In principle one could define $\mathring{f} = \dot{f} + \alpha^i f_{,i}$, α^i being an objective vector, and develop the corresponding more general theory. Owing to the purely academic character of letting $\alpha^i \neq 0$, we confine our attention to $\alpha^i = 0$.

which proves that the material time derivative is nonobjective. On account of (11), the requirement of objectivity for the spot derivative, that is,

$$\mathring{A}'^{i} = Q^{i}{}_{j}\mathring{A}^{j}$$

and the representation of $\mathring{A}'^{i}$ in the new frame $\mathscr{F}'$,namely

$$\mathring{A}'^{i} = \dot{A}'^{i} + H'^{i}{}_{j}A'^{j} + S'^{i}{}_{j}A'^{j}$$

lead us to the transformation law

$$H'^{i}{}_{j} = Q^{i}{}_{p}Q_{j}{}^{q}H^{p}{}_{q} - \dot{Q}^{i}{}_{p}Q_{j}{}^{p} \tag{12}$$

for the nonobjective matrix $H^{i}{}_{j}$. The application of the spatial derivative $\partial/\partial x'^{k} = Q_{k}{}^{r}\, \partial/\partial x^{r}$ to (12) yields

$$H'_{ij,k} = Q_{i}{}^{p}Q_{j}{}^{q}Q_{k}{}^{r}H_{pq,r} \tag{13}$$

whereby $H_{pq,r}$ is an objective tensor. Returning now to (12), symmetrization provides the relation

$$H'_{(ij)} = Q_{i}{}^{p}Q_{j}{}^{q}H_{(pq)} \tag{14}$$

which would endow $H_{(pq)}$ with the objective property; to avoid the contradiction between (14) and the splitting (10) we must set $H_{(ij)} = 0$, whence

$$H_{ij} = H_{[ij]} \tag{15}$$

However, skew-symmetric tensors, too, may be objective. In fact, it follows at once from (12) that the difference between two choices of the matrix $H^{i}{}_{j}$ is an objective tensor. In addition there are outstanding physical quantities which are both objective and skew-symmetric. For instance, if ω_{ij} denotes the intrinsic spin of particles in micropolar bodies,[31,32] the skew-symmetric part of the kinematical tensor $d_{ij} = L_{ij} - \omega_{ij}$ is an objective tensor.

Before drawing a few consequences of this general scheme, let us cast an eye over objective derivatives customarily exhibited in the literature. Fixed a particle **X** of the body, look first at the co-rotational derivative d_r/dt at **X**, that is, the derivative with respect to the frame $\mathscr{F}_r$ whose origin is always at **X** and whose angular velocity is exactly the vorticity at **X**.[6] As is well known from rigid kinematics, in such a case

$$\dot{Q}^{i}{}_{j} = -Q^{i}{}_{m}W^{m}{}_{j} \tag{16}$$

[6] Although these frames were introduced first by Zaremba[33] in 1903, the co-rotational derivative is often named after Jaumann.[34]

Define now the co-rotational time derivative in the frame $\mathscr{F}$ by forcing it to be objective, namely

$$d_r A^j/dt = Q_i^{\ j} \, d_r A'^i/dt$$

Then, on account of (11) and (16), we find that

$$d_r A^j/dt = \dot{A}^j - W^j_{\ k} A^k \tag{17}$$

The expression of the derivative of a mixed tensor may be derived straightaway from Eq. (9). It is a routine matter to show that the co-rotational derivative of the metric tensor vanishes identically and then raising and lowering of indices commutes with d_r/dt.

Another way of finding objective time derivatives is as follows. Let A^k be a vector field of weight w under the group of all coordinate transformations. In particular, consider the transformation from the coordinates X^K, in the reference configuration, to the spatial coordinates x^k of a fixed particle; in this instance the components A^K and A^k are related by

$$A^K \, \partial x^k/\partial X^K = J^w A^k \tag{18}$$

where $J = \det(\partial x^k/\partial X^K)$. Since the coordinate X^K does not change in time, application of the material time derivative operator to (18) yields

$$\dot{A}^K \frac{\partial x^k}{\partial X^K} + A^K \frac{\partial \dot{x}^k}{\partial x^s} \frac{\partial x^s}{\partial X^K} = J^w \left(\frac{\partial A^k}{\partial t} + \dot{x}^s \frac{\partial A^k}{\partial x^s} + w A^k \frac{\partial \dot{x}^s}{\partial x^s} \right) \tag{19}$$

where

$$\dot{J} = J(\partial \dot{x}^s/\partial x^s) \qquad \text{and} \qquad \partial \dot{x}^k/\partial X^K = (\partial \dot{x}^k/\partial x^s)(\partial x^s/\partial X^K)$$

(see, e.g., Ref. 2, pp. 46, 48). Define now the convected time derivative as the objective quantity

$$\frac{d_c A^k}{dt} = J^{-w} \frac{\partial x^k}{\partial X^K} \dot{A}^K$$

Then, in view of the result (19), we can write $d_c A^k/dt$ in terms of spatial quantities only, namely

$$\frac{d_c A^k}{dt} = \frac{\partial A^k}{\partial t} + \underset{\dot{\mathbf{x}}}{\mathscr{L}} A^k \tag{20}$$

where, as usual, the Lie derivative $\underset{\dot{\mathbf{x}}}{\mathscr{L}} A^k$ is defined by

$$\underset{\dot{\mathbf{x}}}{\mathscr{L}} A^k = \dot{x}^s \frac{\partial A^k}{\partial x^s} - A^s \frac{\partial \dot{x}^k}{\partial x^s} + w A^k \frac{\partial \dot{x}^s}{\partial x^s}$$

The extension of (20) to the case of mixed tensors is obvious. As an inner consistency check, we should verify whether the expression (20) transforms tensorially under the Euclidean group (1). A direct calculation shows that such is indeed the case.

We end this section by framing these conclusions within the general scheme outlined above. First, different choices of $H^i{}_j$ and $S^i{}_j$ lead to different objective time derivatives. Two cases in point are the co-rotational and the convected time derivatives which correspond to $H^i{}_j = -W^i{}_j$, $S^i{}_j = 0$ and $H^i{}_j = -W^i{}_j$, $S^i{}_j = -D^i{}_j + wD^k{}_k\delta^i{}_j$, respectively. Hence their difference involves the objective quantities $H^i{}_j = 0$, $S^i{}_j = -D^i{}_j + wD^k{}_k\delta^i{}_j$ (cf. Ref. 11, §151). On the other hand, it is obvious that, given an objective time derivative, the addition of terms depending on the stretching matrix $D^i{}_j$ provides objective time derivatives again (cf. Ref. 2, p. 87). These observations could suggest that every objective time derivative may be constructed by starting from a suitable co-rotational derivative and by adding objective terms. Should this be the case, there would exist a nonobjective vector n^i such that $H_{ij} = n_{[i,j]}$. Of course, this leads unavoidably to the condition $H_{[ij,k]} = 0$, which, according to (13), is an overly restrictive requirement on the objective quantity $H_{ij,k}$ in that we can choose a skew-symmetric matrix H_{ij} satisfying $H_{[ij,k]} \neq 0$ and hence not corresponding to any co-rotational time derivative.

4. RELATIVISTIC OBJECTIVE TIME DERIVATIVES

Relativity does not exhibit any counterpart of the classical Euclidean group of transformations. Then, according to the essence of the epistemological principle of Section 2, in order to describe the behavior of a continuous body, we have to choose a suitable material triad which, in turn, implies the choice of a transport law. Again, look at a general formulation of the problem in such a way that known results are embodied as particular cases.

As pointed out in Section 2, the measuring operations associated with the epistemological principle demand that the constitutive equations of a continuous body are relations between space tensors. On the other hand, as soon as realistic enough descriptions of materials are considered, constitutive equations involve time derivatives of some material functions. Accordingly, any objective time derivative, when acting on space tensors, must preserve the complete orthogonality to the four-velocity field u^α of the body. In the Appendix we derive the structure of time derivatives preserving the spatial character of tensors. So we are led to introduce relativistic spot derivatives defined on scalars and vectors as

$$\begin{aligned} \mathring{f} &= \dot{f} \\ \mathring{A}^\alpha &= \dot{A}^\alpha + A^\beta(\dot{u}^\alpha u_\beta - u^\alpha \dot{u}_\beta + K^\alpha{}_\beta) \end{aligned} \tag{21}$$

where $\dot{A}^\alpha = A^\alpha{}_{;\beta}u^\beta$ and $K^\alpha{}_\beta$ is an arbitrary space tensor ($K^\alpha{}_\beta u_\alpha = 0$, $K^\alpha{}_\beta u^\beta = 0$). For a mixed tensor $A^{\alpha\cdots}{}_{\beta\ldots}$ we find the relation

$$\mathring{A}^{\alpha\cdots}{}_{\beta\ldots} = \dot{A}^{\alpha\cdots}{}_{\beta\ldots} + A^{\gamma\cdots}{}_{\beta\ldots}(\dot{u}^\alpha u_\gamma - u^\alpha \dot{u}_\gamma + K^\alpha{}_\gamma) + \cdots - A^{\alpha\cdots}{}_{\gamma\ldots}(\dot{u}^\gamma u_\beta - u^\gamma \dot{u}_\beta + K^\gamma{}_\beta) - \cdots \tag{22}$$

No matter how the objectivity is set up, the objective time derivatives belong to the class $\mathscr{T}$ of time derivatives (21). In other words, any form of the objectivity principle ultimately results in the selection of a distinguished element (or subclass) of $\mathscr{T}$. Consistently with this assertion, commonly utilized objective time derivatives must be elements of $\mathscr{T}$. In fact, one glance at the literature allows us to say that such is the case. Precisely Oldroyd,[20] Carter and Quintana,[35] and Maugin[12] adopt the derivative

$$\mathring{A}^\alpha = \dot{A}^\alpha - u^\alpha A^\beta \dot{u}_\beta - u^\alpha{}_{;\beta}A^\beta \tag{23}$$

usually termed convected time derivative or projected Lie derivative. In other papers[36] Maugin considers the convective time derivative

$$\mathring{A}^\alpha = \dot{A}^\alpha - u^\alpha A^\beta \dot{u}_\beta - u^\alpha{}_{;\beta}A^\beta + A^\alpha u^\beta{}_{;\beta} \tag{24}$$

Also Lianis' investigation,[22] involving a triad undergoing a Fermi–Walker transport

$$\mathring{A}^\alpha = \dot{A}^\alpha - u^\alpha A^\beta \dot{u}_\beta \tag{25}$$

shows that the compliance with his objectivity principle leads to the form (23) for an absolute vector and to the form (24) for a vector density of weight +1.

In passing, we note that the time derivatives (23)–(25) correspond to setting

$$K^\alpha{}_\beta = -u^\alpha{}_{;\gamma}h^\gamma{}_\beta\,, \qquad K^\alpha{}_\beta = -u^\alpha{}_{;\gamma}h^\gamma{}_\beta + u^\gamma{}_{;\gamma}h^\alpha{}_\beta\,, \qquad K^\alpha{}_\beta = 0$$

respectively, where $h^\alpha{}_\beta = \delta^\alpha{}_\beta + u^\alpha u_\beta$ is the spatial projector (spatial metric).

Within the class $\mathscr{T}$, a particular subclass is recommended on geometrical grounds. The outstanding property of such a subclass is that its elements commute with the process of raising and lowering spatial tensor indices. Then the subclass turns out to be characterized by the requirement

$$\mathring{h}^\alpha{}_\beta = 0 \tag{26}$$

In view of the identity

$$\mathring{h}_{\alpha\beta} = u_\alpha h_{\gamma\beta}\dot{u}^\gamma + u_\beta h_{\alpha\gamma}\dot{u}^\gamma$$

relations (22), (26) provide

$$K_{(\alpha\beta)} = 0 \tag{27}$$

that is, $K_{\alpha\beta} = K^{\gamma}{}_{\beta} h_{\alpha\gamma}$ must be skew-symmetric. In particular, the ansatz

$$K_{\alpha\beta} = -h_{\alpha}{}^{\sigma} h_{\beta}{}^{\rho} u_{[\sigma;\rho]} \equiv -W_{\alpha\beta}$$

plays a prominent role. In fact, the derivative coincides with the time derivative investigated by Estabrook and Wahlquist(37) and by Massa (38) in connection with the topic of physical frame of reference in general relativity. Furthermore, this derivative turns out to be the strict relativistic counterpart of the classical co-rotational derivative in that a spatial triad $e^{\alpha}{}_{(i)}$ propagating along the streamlines according to the law

$$\mathring{e}^{\alpha}{}_{(i)} = \dot{e}^{\alpha}{}_{(i)} - u^{\alpha} e^{\beta}{}_{(i)} \dot{u}_{\beta} - W^{\alpha}{}_{\beta} e^{\beta}{}_{(i)} = 0$$

rotates at the angular velocity of the continuum relative to the compass of inertia.

5. CONCLUSIONS

On the basis of the preceding analysis, in this section we summarize our general viewpoint about objective time derivatives without distinguishing classical and relativistic ones.

From a purely mathematical standpoint, we cannot prefer a particular derivative to others on any account. Special physical circumstances, however, may make one or another derivative appear preferable. The literature concerning involved mechanical behavior of continua bears evidence of evolution equations where a suitable objective time derivative replaces the usual material time derivative. Within the classical context this happens typically in connection with plasticity. More specifically, in 1952 Truesdell(39) carried out a thorough investigation of the behavior of hypoelastic bodies characterized by a stress evolution of the form

$$\frac{d_c}{dt} T^i{}_j = f^i{}_j(D^k{}_h, T^k{}_h)$$

Subsequently, researchers considered the use of the co-rotational derivative d_r/dt, instead of d_c/dt, to be preferable for a sounder theory of plasticity. For example, the co-rotational derivative is customarily considered by Tokuoka(40) and by Mandel(41) and his co-workers. It seems, however, that no crucial argument substantiates these choices. Then, especially in connection with the topic of material with hidden variables[7]—governed by evolution equations involving first-order time derivatives—the choice of the appropriate (objective) time derivative plays a central role. Meanwhile, one may ask whether the answer is universal or depends on the continuum at hand. As purely theoretical considerations are not conclusive, the best thing we can do is to derive consequences from the various choices which eventually allow us to select relevant evolution laws of hidden variables. In this connection, noticeable possibilities are suggested by the analysis of wave propagation. As one might expect, the speeds of propagation and the relations between the amplitudes must change somewhat when we replace a time derivative with another one. This fact is examined in detail in Ref. 42, where we emphasize the new effects of the adoption of the co-rotational derivative in place of the material time derivative.

6. COMMENTS

Within a purely mechanical classical context, objectivity finds its rigorous mathematical statement through invariance under the Euclidean group. Troubles arise when dealing with the interaction of the electromagnetic field with matter because this field is not covariant even under the Galilean group. In order to avoid such troubles, sometimes objectivity is stated in terms of time-independent rotations only.[43]

As a matter of fact, classical objectivity implies that the spin tensor cannot enter into constitutive equations because of its non-Euclidean invariance. On the other hand, this dependence is really required for a rational interpretation of certain phenomena of coupling between electromagnetism and continuum mechanics. For instance, the explanation of the Barnett and Einstein–de Haas effects relies on the assumption of a linear dependence of the free energy function on the spin tensor.[44] Accordingly, it seems that classical objectivity does not work when electromagnetic effects are involved. This should not surprise us at all because electromagnetism is covariant under the Lorentz group, so that great difficulties arise when rigid motions are concerned.[45] To overcome this unpleasant feature, our attention could be fixed on the relativistic context but, unfortunately, there we will find more than one definition of objectivity.

Once again, the validity of principles, such as objectivity, ought to be checked by experiment. If tests of some materials[8] show that the principle

[7] See, e.g., Refs. 8, 9.

[8] Difficulties arising in conjunction with comparison between theory and experiment are outlined in Ref. 46.

of objectivity is not valid in certain circumstances, then they are far better disproof of the principle than any mathematical result.[15] Yet, beyond experimental checks, doubts motivated on theoretical grounds may be cast upon the classical formulation of the principle of objectivity.[47] Also, if within a self-contained theory like relativity there is no natural setting for an objectivity principle coming out directly from genuine relativistic requirements, then we are led to suspect that the classical realization is merely a fortuitous coincidence.

APPENDIX

For any space vector S^α $(S^\alpha u_\alpha = 0)$ the most general time derivative preserving the spatial character is obviously given by

$$\mathring{S}^\alpha = h^\alpha{}_\beta \dot{S}^\beta + K^\alpha{}_\beta S^\beta \equiv \dot{S}^\alpha - u^\alpha S^\beta \dot{u}_\beta + K^\alpha{}_\beta S^\beta \tag{A1}$$

where $\dot{S}^\beta = S^\beta{}_{;\gamma} u^\gamma$, $K^\alpha{}_\beta$ is an arbitrary space tensor $(K^\alpha{}_\beta u^\beta = 0,\ K^\alpha{}_\beta u_\alpha = 0)$, and $h^\alpha{}_\beta = \delta^\alpha{}_\beta + u^\alpha u_\beta$ is the spatial projector. As usual, we set

$$\mathring{f} = \dot{f} \tag{A2}$$

for any scalar f. So as to extend (A1) to arbitrary vectors we need the definition of $\mathring{u}^\alpha$. Now, since $\mathring{S}^\alpha u_\alpha = 0$, we have

$$0 = (S^\alpha u_\alpha)^\circ = S^\alpha \mathring{u}_\alpha$$

whereby

$$\mathring{u}_\alpha = \lambda u_\alpha \tag{A3}$$

for some λ. For the sake of definiteness we assume that u_α undergoes a parallel transport, whence $\lambda = 0$. In such a case the spot derivatives commute with the process of raising and lowering spatial tensor indices if and only if they do so for spacetime tensor indices.

Look at a triad of space vectors $e^{(i)}_\alpha$ subject to the transport law (A1), i.e., $\mathring{e}^{(i)}_\alpha = 0$. Set $e^{(0)}_\alpha = u_\alpha$. So we may write

$$\dot{e}^{(\mu)}_\alpha = (u_\alpha \dot{u}^\beta - \dot{u}_\alpha u^\beta + K^\beta{}_\alpha)\, e^{(\mu)}_\beta \tag{A4}$$

For any vector A^α, (A2) yields

$$(A^\alpha e^{(\mu)}_\alpha)^\circ = (A^\alpha e^{(\mu)}_\alpha)^\cdot$$

Hence, on account of (A3), (A4), we find the relation (21) of the text.

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