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VISCOUS FLUIDS WITH HIDDEN VARIABLES AND HYPERBOLIC SYSTEMS

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The behaviour of viscous fluids is described through a hidden variable approach which leads to a hyperbolic quasi-linear system. Such a system accounts even for transverse weak discontinuities which are shown to be exceptional waves.

1. Introduction

Owing to the extensive literature that appeared after the pioneering paper of Coleman and Gurtin [1] in 1967 we have now become accustomed to the use of hidden variables in continuum mechanics. In essence a material with hidden variables consists of a set of response functions supplemented by first-order ordinary differential equations governing the evolution of a suitable set of variables which account for internal (hidden) degrees of freedom of the material at hand. The hidden variable approach turned out to be very fruitful in various contexts—as, for example, in the theory of wave propagation in heat conductors elaborated by Kosiński and Perzyna [2]— and it has been given a formal mathematical structure in a paper of Day [3].

Encouraged by the great flexibility displayed by the model and motivated by the need for a satisfactory account of viscosity, recently one of us [4] attempted to remove, with recourse to hidden variables, the paradox of infinite wave speed in viscous materials. Although this research resulted in a consistent scheme of jump relations for strong and weak discontinuities, we still need a comprehensive analysis of the hyperbolicity condition of the associated quasi-linear system of first-order equations. It is just the aim of the present note to remedy this deficiency by proving that the model of fluids with hidden variables [4] gives rise to a hyperbolic quasi-linear system.

In Section 2 we set up the basic equations describing the behaviour of viscous fluids with hidden variables. Next, following throughout the standard procedure [5, 6], we test the hyperbolicity of the quasi-linear system and derive the full set of characteristic speeds (Section 3). The main result to emerge from this note is the existence of transverse waves which, in Section 4, are shown to be exceptional waves.

Notations. Henceforth ρ stands for the mass density, θ the temperature, e the internal energy, η the entropy, $\psi = e - \theta\eta$ the free energy, $\boldsymbol{T}$ the stress, $\boldsymbol{v}$ the velocity, $\boldsymbol{D}$ the symmetric gradient of velocity, ∇ the spatial gradient operator. A superposed dot denotes the material time derivative, a comma partial differentiation; so, for any function ξ, $\dot{\xi} = \xi_{,t} + \boldsymbol{v} \cdot \nabla\xi$. The subscripts ρ, θ denote partial differentiations. Capital latin indices run from 1 to 11 while small latin indices run from 1 to 3; repeated indices imply summation.

2. The quasi-linear system

Following along the general lines presented in [4] the behaviour of a viscous fluid is taken to be expressed through a set of C^2 response functions $\sigma = \sigma(\theta, \rho, \boldsymbol{\alpha})$, where $\sigma = \psi, \eta, \boldsymbol{T})$, and through the linear differential equation

$$\dot{\boldsymbol{\alpha}} = (\boldsymbol{D} - \boldsymbol{\alpha})/\tau \tag{1}$$

governing the evolution of the hidden variable $\boldsymbol{\alpha}$ whose values are in the space of symmetric tensors. According to (1) the parameter $\tau > 0$ plays the role of the relaxation time.

The thermodynamic analysis carried out in [4] shows that, letting the free energy function ψ be expressed as

$$\psi = \Psi(\theta, \rho) + (\tau/\rho)\{\mu\boldsymbol{\alpha} : \boldsymbol{\alpha} + \tfrac{1}{2}\lambda(\mathrm{tr}\,\boldsymbol{\alpha})^2\},$$

compatibility with the second law of thermodynamics—in the form of the Clausius–Duhem inequality—yields the restrictions

$$\mu \geqslant 0, \qquad 3\lambda + 2\mu \geqslant 0 \tag{2}$$

and the constitutive relations

$$\eta = -\Psi_\theta, \qquad \boldsymbol{T} = -p\boldsymbol{I} + 2\mu\boldsymbol{\alpha} + \lambda(\mathrm{tr}\,\boldsymbol{\alpha})\boldsymbol{I}, \tag{3}$$

where $p = \rho^2\psi_\rho$. It is a simple matter to see that if $\boldsymbol{D}$ is time independent the hidden variable $\boldsymbol{\alpha}$ asymptotically becomes $\boldsymbol{D}$ itself. This fact assigns to the parameters μ, λ the role of the usual viscosity coefficients.

Fix an orthonormal frame $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, and now introduce the array

$$u^A = (\rho, v_3, v_1, v_2, \theta, \alpha_{33}, \alpha_{11}, \alpha_{22}, \alpha_{13}, \alpha_{23}, \alpha_{12}).$$

Then upon substitution of (1), (3) into the customary balance equations,

$$\dot{\rho} + \rho\nabla \cdot \boldsymbol{v} = 0, \qquad \rho\dot{\boldsymbol{v}} - \nabla \cdot \boldsymbol{T} = 0, \qquad \rho\dot{e} - \boldsymbol{T} : \boldsymbol{D} = 0,$$

the equations which result and (1) may be cast in the form of a quasi-linear system of the first-order

$$u^A_{,t} + (a^i)^A_B u^B_{,i} + b^A = 0, \tag{4}$$

where

$$b^A = \left[0, 0, 0, 0, -(\rho\theta\eta_\theta)^{-1}\{2\mu\boldsymbol{\alpha} : \boldsymbol{\alpha} + \lambda(\mathrm{tr}\,\boldsymbol{\alpha})^2\}, \frac{\alpha_{33}}{\tau}, \frac{\alpha_{11}}{\tau}, \frac{\alpha_{22}}{\tau}, \frac{\alpha_{13}}{\tau}, \frac{\alpha_{23}}{\tau}, \frac{\alpha_{12}}{\tau}\right].$$

For the sake of conciseness we do not write the explicit form of the matrices a^i which may be derived straightaway.

3. Hyperbolicity of the quasi-linear system

Let $f(x^i, t) = 0$ be a surface in space-time and denote by $\boldsymbol{n} = \nabla f/|\nabla f|$ its unit normal. For $f = 0$ to be a characteristic surface f must satisfy the determinantal equation

$$|(a^i)^A_B f_{,i} + \delta^A_B f_{,t}| = 0.$$

For ease in writing set

$$\varepsilon_1 = -(2\mu\tau/\rho)\alpha_{11} - (\lambda/\rho)(1+\tau \operatorname{tr} \boldsymbol{\alpha}), \qquad \varepsilon_2 = -(2\mu\tau/\rho)\alpha_{22} - (\lambda/\rho)(1+\tau \operatorname{tr} \boldsymbol{\alpha}),$$

$$\varepsilon_3 = -(2\mu/\rho)(1+\tau\alpha_{33}) - (\lambda/\rho)(1+\tau \operatorname{tr} \boldsymbol{\alpha}), \qquad v_n = \boldsymbol{v} \cdot \boldsymbol{n};$$

moreover, without loss of generality, assume that $\boldsymbol{e}_3 = \boldsymbol{n}$. In accordance with the general theory [5, 6] we say that the system (4) is hyperbolic if the matrix

$$(a^i)^A_B n_i = \begin{pmatrix}
v_n & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_\rho/\rho & v_n & 0 & 0 & p_\theta/\rho & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & -\frac{4\mu\tau}{\rho}\alpha_{13} & -\frac{4\mu\tau}{\rho}\alpha_{23} & -\frac{4\mu\tau}{\rho}\alpha_{12} \\
0 & 0 & v_n & 0 & 0 & 0 & 0 & 0 & -2\mu/\rho & 0 & 0 \\
0 & 0 & 0 & v_n & 0 & 0 & 0 & 0 & 0 & -2\mu/\rho & 0 \\
0 & -\rho\eta_\rho/\eta_\theta & 0 & 0 & v_n & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/\tau & 0 & 0 & 0 & v_n & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & v_n & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & v_n & 0 & 0 & 0 \\
\theta & 0 & -1/2\tau & 0 & 0 & 0 & 0 & 0 & v_n & 0 & 0 \\
0 & 0 & 0 & -1/2\tau & 0 & 0 & 0 & 0 & 0 & v_n & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_n
\end{pmatrix}$$

admits a complete real set of (i.e. eleven linearly independent) eigenvectors. As a matter of fact, a direct calculation provides the following eigenvalues:

$$c_0 = v_n, \quad m = 5; \qquad c_T^+ = v_n + (\mu/\rho\tau)^{1/2}, \quad m = 2; \qquad c_T^- = v_n - (\mu/\rho\tau)^{1/2}, \quad m = 2;$$

$$c_L^+ = v_n + \left\{\hat{p}_\rho + \frac{1}{\rho}\left(\frac{2\mu+\lambda}{\tau} + 2\mu\boldsymbol{n}\cdot\boldsymbol{\alpha}\boldsymbol{n} + \lambda \operatorname{tr} \boldsymbol{\alpha}\right)\right\}^{1/2};$$

$$c_L^- = v_n - \left\{\hat{p}_\rho + \frac{1}{\rho}\left(\frac{2\mu+\lambda}{\tau} + 2\mu\boldsymbol{n}\cdot\boldsymbol{\alpha}\boldsymbol{n} + \lambda \operatorname{tr} \boldsymbol{\alpha}\right)\right\}^{1/2};$$

where m is the multiplicity and $\hat{p}_\rho = p_\rho - p_\theta\eta_\rho/\eta_\theta$ is the derivative of p with respect to ρ at constant entropy. The corresponding eleven eigenvectors turn out to be linearly independent, thus making the system hyperbolic, provided the roots $c_L^\pm$ are real. This certainly happens if $\tau(\boldsymbol{D}:\boldsymbol{D})^{1/2} < 1$; such a condition may be phrased by saying that the characteristic time $(\boldsymbol{D}:\boldsymbol{D})^{-1/2}$ must be greater than the relaxation time τ.

Observe that this sort of restriction on the range of validity of the above model is typical of any macroscopic theory.

Two remarks are now in order. First, as we expect it to happen, in the limiting case of vanishing viscosity only three different eigenvalues occur, namely

$$c_0 = v_n \quad (m=9), \qquad c_L^+ v_n + (\hat{p}_\rho)^{1/2}, \qquad c_L^- = v_n - (\hat{p}_\rho)^{1/2}.$$

Second, the eigenvalues $c_T^\pm$, $c_L^\pm$ tend to infinity as $\tau \to 0$. This feature is consistent with the fact that the case $\tau = 0$ corresponds to Navier–Stokes' law which, as is well known, forbids wave propagation at finite speed.

4. Exceptionality of the transverse waves

In this section we analyze the evolution of the waves propagating at speed $c_T^\pm$ into a constant state; in such a case $\boldsymbol{\alpha} = 0$ [4]. In general every eigenvalue $c_T^\pm$ is associated with two right and left eigenvectors $r_\pm^A$, $l_A^\pm$, namely

$$\begin{aligned}
(r_\pm^A)_1 &= (\pm\rho(\rho\tau/\mu)^{1/2}K, K, 1, 0, \mp(\rho\eta_\rho/\eta_\theta)(\rho\tau/\mu)^{1/2}K, \mp(\rho/\mu\tau)^{1/2}K, 0, 0, \mp\tfrac{1}{2}(\rho/\mu\tau)^{1/2}, 0, 0),\\
(l_A^\pm)_1 &= (0, 0, \tfrac{1}{2}, 0, 0, 0, 0, 0, \mp(\mu\tau/\rho)^{1/2}, 0, 0),\\
(r_\pm^A)_2 &= (\pm\rho(\rho\tau/\mu)^{1/2}H, H, 0, 1, \mp(\rho\eta_\rho/\eta_\theta)(\rho\tau/\mu)^{1/2}H, \mp(\rho/\mu\tau)^{1/2}H, 0, 0, 0, \mp\tfrac{1}{2}(\rho/\mu\tau)^{1/2}, 0),\\
(l_A^\pm)_2 &= (0, 0, 0, \tfrac{1}{2}, 0, 0, 0, 0, 0, \mp(\mu\tau/\rho)^{1/2}, 0),\\
K &= 2\tau\{1-(\rho\tau/\mu)(\hat{p}_\rho + ((2\mu+\lambda)/\tau + 2\mu\boldsymbol{n}\cdot\boldsymbol{\alpha}\boldsymbol{n} + \lambda\ \mathrm{tr}\,\boldsymbol{\alpha})/\rho)\}^{-1}\alpha_{13},\\
H &= 2\tau\{1-(\rho\tau/\mu)(\hat{p}_\rho + ((2\mu+\lambda)/\tau + 2\mu\boldsymbol{n}\cdot\boldsymbol{\alpha}\boldsymbol{n} + \lambda\ \mathrm{tr}\,\boldsymbol{\alpha})/\rho)\}^{-1}\alpha_{23},
\end{aligned}$$

where the arbitrary constants are so chosen as to make the orthonormality condition true. Here, the subscripts 1, 2 distinguish the two propagation modes corresponding to the possible polarizations of the transverse waves at hand. Letting l_A, r^A correspond to the same propagation mode, the rays for the system (4) are the curves $x^i = x^i(t)$ defined by [5, 6] $\mathrm{d}x^i/\mathrm{d}t = \lambda^i$ with $\lambda^i = l_A(a^i)_B^A r^B$.

The general relation governing the growth of a weak discontinuity shows that the exceptionality of the wave is ensured by the vanishing along the rays of the quantity [6, 7]

$$N = \{D_Q(l_A(a^i)_B^A)\}_0 f_{,i} r_0^Q r_0^B + \{D_Q(l_B)\}_0 f_{,t} r_0^Q r_0^B,$$

where a subscript $_0$ indicates that the corresponding quantities are evaluated in the unperturbed state and D_Q denotes the derivative operator with respect to the field variables u^Q. In the present case, owing to the transversality of the wave, a straightforward calculation yields the result $N=0$ which implies that the transverse waves are exceptional.

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References

[1] B.D. Coleman and M.E. Gurtin, "Thermodynamics with internal state variables", *J. Chem. Phys. 47*, 597 (1967).
[2] W. Kosiński and P. Perzyna, "Analysis of acceleration waves in material with internal parameters", *Arch. Mech. 24*, 629 (1972).
[3] W.A. Day, "Entropy and hidden variables in continuum thermodynamics", *Rational Mech. Anal. 62*, 367 (1976).
[4] A. Morro, "Wave propagation in thermo-viscous materials with hidden variables", *Arch. Mech. 32*, 145 (1980).
[5] A. Jeffrey, *Quasi-linear Hyperbolic Systems and Waves*, Pitman, London (1976).
[6] C. Cattaneo, lectures given at the "Istituto Nazionale di Alta Matematica", 1978. A brief account of the theory is given in: A.M. Anile, G. Mulone, and S. Pluchino, "Critical time for shock formation in radiative magnetogasdynamics", *Wave Motion 1*, 163 (1979).
[7] A. Jeffrey, "Nonlinear wave propagation", *Z. Angew. Math. Mech. 58*, T38 (1978).