Korteweg-de Vries Equation and Nonlinear Waves.

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In recent years much attention has been drawn to the Korteweg-de Vries equation (KdV); in this respect an exhaustive account is exhibited in the review article by MIURA (1). At least two reasons explain the importance of the KdV. First, the KdV turns out to be a significant approximation in a large class of physical problems like, for example, ion-acoustic waves in plasma, magneto-acoustic waves in plasma, the anharmonic lattice, longitudinal dispersive waves in elastic rods, thermally excited phonon packets in low-temperature nonlinear crystals (2). Second, the KdV admits particular solutions, termed solitons, whose exciting properties determine an increasing interest in the literature (3). In spite of being widely investigated, in our opinion the KdV deserves further attention as to its physical relevance especially in connection with the earliest problem of water wave propagation. Such an analysis is in order particularly because the various derivations of the KdV look as formal derivations where the mathematical aspects hide the physical ones.

It is the purpose of this note to shed light on the physical approximations which are at the basis of the KdV and, meanwhile, to show an approach like that of SU and GARDNER (4) allows us to give a precise interpretation of the procedure followed by Korteweg and de Vries themselves (5). The starting point is the set of equations attained by Green and Naghdi (6); this avoids *ad hoc* corrections to the usual shallow water theory (4) or formal expansions with respect to the vertical co-ordinate (5,7).

By analogy with Korteweg and de Vries' paper, try to find the evolution equation of a solitary wave propagating along a canal with a flat bottom. To this purpose we make the guess that the behaviour of the actual wave may be expressed through a

(1) R. M. MIURA: *SIAM Rev.*, **18**, 412 (1976).
(2) See, *e.g.*, A. C. SCOTT, F. Y. F. CHU and D. W. MCLAUGHLIN: *Proc. IEEE*, **61**, 1443 (1973) and references cited therein.
(3) K. LONNGREN and A. C. SCOTT: *Solutions in Action* (New York, N. Y., 1978).
(4) C. H. SU and C. S. GARDNER: *J. Math. Phys.* (*N. Y.*), **10**, 536 (1969).
(5) D. J. KORTEWEG and G. DE VRIES: *Philos. Mag.*, **39**, 422 (1895).
(6) A. E. GREEN and P. M. NAGHDI: *J. Fluid Mech.*, **78**, 237 (1976).
(7) A. JEFFREY and T. KAKUTANI: *SIAM Rev.*, **14**, 582 (1972). See also G. B. WHITHAM: *Linear and Nonlinear Waves* (New York, N. Y., 1974). There the derivation of the KdV is based on considerations about dispersion.

suitable correction of the fundamental wave governed by the differential equation

$$\varphi_t + c_0 \varphi_\chi = 0\,, \qquad c_0 = (g h_0)^{\frac{1}{2}}\,,$$

where $\varphi(\chi, t)$ is the actual height of the fluid in the place χ at time t, h_0 is the equilibrium height and subscripts denote partial differentiation. Accordingly it is convenient to work in the frame of reference (x, t) at rest with respect to the fundamental wave, namely

$$x = \chi - c_0 t\,, \qquad t = t\,.$$

Borrowing from Su and Gardner we account for long waves slowly varying in time by introducing the ordering parameter ε and the new variables

$$\xi = \varepsilon^{\frac{1}{2}} x\,, \qquad \tau = \varepsilon^{\frac{3}{2}} t\,. \tag{1}$$

The parameter ε, which can be thought of as the ratio of the wave amplitude to h_0, lends a precise meaning to the different weight of the derivatives under consideration; for example, $\varphi_x = O(\varepsilon^{-\frac{1}{2}})$ while $\varphi_t = O(\varepsilon^{-\frac{3}{2}})$. Moreover it is worth remarking that the choice of the exponents $\frac{1}{2}$ and $\frac{3}{2}$ in (1) is the only one leading to reasonable results [8].

The behaviour of the fluid is assumed to be described by Green and Naghdi's equations which may be cast in the form

$$\varphi_t + (u\varphi)_\chi = 0\,, \qquad u_t + u u_\chi + \varphi^{-1} P_\chi = 0\,, \tag{2}$$

where u is the horizontal component of the velocity and $P = g\varphi^2/2 + \varphi^2 \ddot{\varphi}/3$ [9]. In terms of ξ, τ eqs. (2) can be written as

$$\varepsilon\varphi_\tau + (u - c_0)\varphi_\xi + \varphi u_\xi = 0\,, \qquad \varepsilon u_\tau + (u - c_0) u_\xi + \varphi^{-1} P_\xi = 0\,.$$

Then, in view of the formal expansion of φ, u around the equilibrium values h_0, 0, that is

$$\varphi = h_0 + \varepsilon\varphi' + \varepsilon^2\varphi'' + \dots\,, \qquad u = \varepsilon u' + \varepsilon^2 u'' + \dots\,,$$

in the first order of approximation we find that

$$u'_\xi = \frac{c_0}{h_0}\varphi'_\xi\,,$$

whence

$$u' = \frac{c_0}{h_0}(\varphi' + \beta)\,. \tag{3}$$

From a purely mathematical point of view β is an arbitrary function of time. To get a strict connection with Korteweg and de Vries' procedure we let β be a small arbitrary

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(8) A. JEFFREY: *Z. Angew. Math. Mech.*, **58**, T38 (1978).
(9) F. BAMPI and A. MORRO: *Nuovo Cimento, C*, **1**, 377 (1978).

constant though at least two reasons suggest to set $\beta = 0$ [10]. Then, on account of (3), the next order of approximation and some algebraic manipulations yield the desired equation

$$\varphi'_\tau + \frac{3c_0}{2h_0}\varphi'\varphi'_\xi + \frac{c_0}{h_0}\beta\varphi'_\xi + \frac{1}{6}c_0 h_0^2 \varphi'_{\xi\xi\xi} = 0 \tag{4}$$

which reduces exactly to the original form of the KdV in terms of x, t once we set $\varepsilon\varphi' = \eta$ and $\varepsilon\beta = \alpha$.

Some remarks are now in order about peculiar aspects of the KdV. Observe first that, although any differential equation formally equivalent to (4) is acceptable in a mathematical context, the same is not true in a physical context because the significant unknown function is fixed at the outset while the independent variables are x, t or any pair of co-ordinates related to x, t by a Galileian transformation. Among the Galilean frames, (x, t) and (χ, t) appear to be clearly privileged. In such frames eq. (4) becomes

$$\eta_t + \frac{3c_0}{2h_0}\eta\eta_x + \frac{c_0}{h_0}\alpha\eta_x + \frac{1}{6}c_0 h_0^2 \eta_{xxx} = 0 \tag{5}$$

and

$$\eta_t + \frac{3c_0}{2h_0}\eta\eta_\chi + \frac{c_0}{h_0}(h_0 + \alpha)\eta_\chi + \frac{1}{6}c_0 h_0^2 \eta_{\chi\chi\chi} = 0\,, \tag{6}$$

respectively. In so doing we take it that η is invariant under Galilean transformations; this requirement is consistent wich the properties of the starting equations (2) which have been derived from the energy balance just through the use of Galilean invariance. On the contrary, Galilean invariance cannot hold for some alternatives to the KdV, namely Benjamin-Bona-Mahony and Jeffrey equations [11], concerning wave motion in fluids.

Often it is claimed that an expression of the KdV like (6) may be achieved by applying to (5) the transformation

$$x \to \chi\,, \qquad t \to t\,, \qquad \eta \to \eta + \tfrac{2}{3}h_0\,.$$

This emphasises how the counterpart of a non-Galilean transformation, that is the replacement of x with χ, is the introduction of the unphysical quantity $\eta + 2h_0/3$ [1].

Finally, look at the term $\eta_{\chi\chi\chi}$ which is responsible for dispersion of waves and, in the present context, it follows directly from the term $\varphi^2\ddot{\varphi}$ occurring in Green and Naghdi's theory. More precisely, the dispersive term $\eta_{\chi\chi\chi}$ is originated by the gradient of the vertical acceleration, $\ddot{\varphi}_\chi$, thereby showing that the dispersive effects arise as soon as the theory accounts adequately for the vertical motion of the fluid particles. This observation makes it obvious the fact that the shallow water theory, disregarding the vertical acceleration, is not a proper starting point for deriving the KdV as it appears in Su and Gardner's paper [4].

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[10] F. Bampi and A. Morro: *Water wave theories and variational principles*, submitted for publication.
[11] A. Jeffrey: *Applicable Analysis*, **7**, 159 (1978). See also ref. [10].