Water Wave Theories and Variational Principles.

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Summary. — A unified account of the most outstanding models for water wave propagation is given. A first scheme assembles the shallow-water equations, the Boussinesq equations and the Korteweg-de Vries equation as particular cases of the Green and Naghdi equations. Instead, Benjamin-Bona-Mahony's and Jeffrey's equations cannot be set in such a scheme: this is ascribed to their non-Galileian invariance. A second scheme emphasizes the different peculiarities of the various models through the corresponding variational formulations. In this context a variational principle for Green and Naghdi's equations is set up.

1. – Introduction.

The aim of the present paper is twofold: to provide a unified account of the most outstanding water wave models and to inspect their variational counterparts.

These purposes are to be viewed in conjunction with the fact that a general theory, allowing for the nonlinear inertia terms and the nonlinear boundary condition over an unknown surface (1), turns out to be of little handiness for practical problems. To overcome such a difficulty several approximate models appeared in the literature; among them we cite the shallow-water theory (2) and the equations of Boussinesq (3), Korteweg-de Vries (4), Benjamin-Bona-

(1) F. BAMPI and A. MORRO: *Nuovo Cimento C*, **1**, 377 (1978).
(2) J. J. STOKER: *Water Waves* (New York, N. Y., 1957).
(3) J. BOUSSINESQ: *C. R. Acad. Sci.*, **72**, 755 (1871).
(4) D. J. KORTEWEG and G. DE VRIES: *Philos. Mag.*, **39**, 422 (1895).

Mahony [5] and Jeffrey [6]. Needless to say, a proper scheme gathering these models would be profitable both on theoretical and experimental grounds. Here such a scheme is displayed through Green and Naghdi's theory [7] by showing how the various models can be derived from it.

It is a general feature of a variational formulation that, besides giving new insights into the related theory, it provides a comprehensive synthesis of the theory itself. Further, variational formulations may be used as a basis for numerical computations such as finite and infinite elements [8]. In spite of this, a search for variational formulations in hydrodynamics appears to be a very hard task merely for lack of a systematic method determining the Lagrangian density. This point is emphasized in the present paper where we re-examine the general aspects of a variational formulation and, by means of a customary procedure, we get a variational principle for Green and Nadhdi's model.

In summary, the plan of the paper is as follows. Starting from Green and Naghdi's equations, sect. **2** exhibits the shallow-water theory and the equations of Boussinesq and Korteweg-de Vries as particular cases. The alternatives to the Korteweg-de Vries equation, namely the Benjamin-Bona-Mahony equation and the Jeffrey equation, are examined in sect. **3**. Such alternatives appear not to be particular cases of Green and Naghdi's model and they turn out not to be Galileian invariant: it is conjectured that these properties are each other closely related. The account of the models is then improved by considering their variational counterparts. To this end, sect. **4** deals with general remarks about the mathematical procedures which distinguish the role played by the unknown function (velocity potential) from that of the variable domain (free surface). The problem becomes particularly simple (fixed domain) in connection with the known approximate models (sect. **5**). In the case of more involved models, such as Green and Naghdi's model, a variational formulation may be achieved by using the velocity potential in the Clebsch form (sect. **6**).

Notations. The fluid is moving between the bottom $\boldsymbol{x} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 - h(x, y)\boldsymbol{e}_3$ and the free surface $\boldsymbol{x} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + \eta(x, y, t)\boldsymbol{e}_3$, where t is the time and (x, y) belongs to a suitable bidimensional region D. A superposed dot denotes the total time derivative, subscripts denote partial derivatives, ∇ is the bidimensional gradient operator, *i.e.* $\nabla \equiv (\partial/\partial x)\boldsymbol{e}_1 + (\partial/\partial y)\boldsymbol{e}_2$. Moreover, $\varphi = \eta + h$, $\psi = (\eta - h)/2$, $\lambda = \dot{\psi}$, $w = \dot{\varphi}$. The velocity of a fluid particle is $\boldsymbol{V} = \dot{\boldsymbol{x}} =$

(5) T. B. Benjamin, J. L. Bona and J. J. Mahony: *Philos. Trans. R. Soc. London Ser. A*, **272**, 47 (1972).
(6) A. Jeffrey: *Z. Angew. Math. Mech.*, **58**, 38 (1978).
(7) A. E. Green and P. M. Naghdi: *J. Fluid Mech.*, **78**, 237 (1976). See also ref. (1).
(8) See, *e.g.*, P. Bettess and O. C. Zienkiewicz: *Int. J. Numer. Methods Eng.*, **11**, 1271 (1977).

$= \boldsymbol{v} + (\lambda + Xw)\boldsymbol{e}_3$, where $X \in [-\frac{1}{2}, \frac{1}{2}]$ is the vertical Lagrangian co-ordinate. The pressure field is $p(\boldsymbol{x}, t)$, while P is the pressure at the bottom, p_a is the atmospheric pressure, and $\Pi = \int_{-h}^{\eta} p \, dz$.

2. – Outstanding water wave theories from Green and Naghdi's.

According to the GN [9] model, the motion of a fluid with constant mass density ϱ is described by the equations

$$(2.1) \qquad \begin{cases} \dot{\varphi} + \varphi \nabla \cdot \boldsymbol{v} = 0 \,, \\ \varrho\varphi\dot{\boldsymbol{v}} = -\nabla\Pi + p_a \nabla\eta + P\nabla h \,, \\ \varrho\varphi\dot{\lambda} = P - p_a - \varrho g \varphi \,, \\ \frac{1}{12}\varrho\varphi^2 \dot{w} = \Pi - \frac{1}{2}(P + p_a)\varphi \,. \end{cases}$$

As will be shown in a moment, the significant special cases of (2.1) may be framed in two main classes. The first class arises out by assuming that the pressure be identified with the hydrostatic pressure (shallow water). The second class concerns models related to flat bottoms, that is $h(x, y) = h_0$.

1) *Shallow-water theory*. The typical assumption of this approximate model can be expressed as

$$p = \varrho g(\eta - z) + p_a \,.$$

In such a case obvious integrations yield

$$P = \varrho g \varphi + p_a \,, \qquad \Pi = \{\tfrac{1}{2}\varrho g \varphi + p_a\}\varphi \,.$$

Substitution in $(2.1)_{3,4}$ gives

$$\dot{\lambda} = 0 \,, \qquad \dot{w} = 0 \,,$$

and hence the vertical acceleration of the particles, $\dot{\lambda} + X\dot{w}$, vanishes identically [1]. Finally, eqs. $(2.1)_{1,2}$ become the usual system of nonlinear shallow-

[9] Henceforth we use the following shorthands: GN for Green-Naghdi, B for Boussinesq, KdV for Korteweg-de Vries, BBM for Benjamin-Bona-Mahony, J for Jeffrey.

water theory, namely

$$\left\{\begin{array}{l}\eta_t+\nabla\cdot\{(\eta+h)\boldsymbol{v}\}=0\,,\\ \boldsymbol{v}_t+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=-\,g\nabla\eta\,.\end{array}\right. \tag{2.2}$$

2) *Flat-bottom theories.* The assumption $h=h_0$ simplifies the GN equations to

$$\left\{\begin{array}{l}\dot{\varphi}+\varphi\,\nabla\cdot\boldsymbol{v}=0\,,\\ \varrho\varphi\dot{\boldsymbol{v}}=-\,\nabla(\Pi-p_a\varphi)\,,\end{array}\right. \tag{2.3}$$

while

$$\Pi-p_a\varphi=\tfrac{1}{3}\varrho\varphi^2\ddot{\varphi}+\tfrac{1}{2}\varrho g\varphi^2\,.$$

Observe that the essential consequence of the flatness assumption is the disappearance of the term $P\,\nabla h$ and this ultimately allows the quantity P to be dropped out from the unknowns of the problem.

i) *Boussinesq equations.* Equations (2.3) may be written in the equivalent form

$$\left\{\begin{array}{l}\varphi_t+\nabla\cdot(\varphi\boldsymbol{v})=0\,,\\ \boldsymbol{v}_t+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=-\,g\,\nabla\varphi-\tfrac{1}{3}\varphi\,\nabla\ddot{\varphi}-\tfrac{2}{3}\,\ddot{\varphi}\,\nabla\varphi\,.\end{array}\right. \tag{2.3$'$}$$

Introduce now the Boussinesq approximation whereby the function φ must appear only through linear terms so that, for instance, $\ddot{\varphi}\simeq\varphi_{tt}$. Accordingly we obtain the B equation in the form

$$\left\{\begin{array}{l}\varphi_t+\nabla\cdot(\varphi\boldsymbol{v})=0\,,\\ \boldsymbol{v}_t+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=-\,g\,\nabla\varphi-\tfrac{1}{3}h_0\,\nabla\varphi_{tt}\,.\end{array}\right. \tag{2.4}$$

Physically, the Boussinesq approximation is tantamount to neglecting the velocity of the fluid particles against the propagation velocity of the surface wave which is approximately equal to $(gh_0)^{\frac{1}{2}}$.

In passing, we note that the linear counterpart of (2.3) leads to

$$\varphi_t+h_0\,\nabla\cdot\boldsymbol{v}=0\,,$$

$$\boldsymbol{v}_t=-\,\nabla\varphi-\tfrac{1}{3}h_0\,\nabla\varphi_{tt}\,,$$

which are just the linear Boussinesq equations.

ii) *Korteweg-de Vries equation.* Look at the unidimensional counterpart of (2.1). Precisely, if we assume that the fields at hand depend only on one spatial co-ordinate, x say, and denote by u the x-component of $\boldsymbol{v}$, eqs. (2.1) can be written as

$$\begin{cases} \varphi_t + (\varphi u)_x = 0 \, , \\ (\varphi u)_t + (\varphi u^2 + \mathscr{P})_x = 0 \, , \end{cases} \tag{2.5}$$

where $\mathscr{P} = \varphi^2 \ddot{\varphi}/3 + g\varphi^2/2$. The structure of the system (2.5) allows us to apply the method of Su and Gardner ([10]) to derive the KdV equation. Here some details of proof are given so as to emphasize the basic approximation related to the KdV equation.

In the linear approximation, (2.5) leads to

$$\varphi_{tt} - g h_0 \varphi_{xx} = 0 \, , \tag{2.6}$$

which accounts for waves travelling to both left and right with speed $c_0 = (gh_0)^{\frac{1}{2}}$. On this observation, consider a wave moving to the right with speed c_0 as fundamental solution of (2.5). Since the actual wave motion shows dispersive effects ([11]), the presence of derivatives of higher order is needed to get a more realistic model. Accordingly, introduce a transformation of the independent variables x, t making $x - c_0 t$ the dominant variable, namely

$$\begin{cases} \xi = \varepsilon^{\alpha}(x - c_0 t) \, , \\ \tau = \varepsilon^{\alpha+1} t \, ; \end{cases} \tag{2.7}$$

the parameter α will be determined later. It is an immediate consequence of (2.7) that the transformation of the speed is given by the relation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c_0 + \varepsilon \frac{\mathrm{d}\xi}{\mathrm{d}\tau} \, ,$$

which clarifies how the parameter ε accounts for the difference between the actual propagation wave and the fundamental one.

The same parameter ε is now adopted as an ordering parameter by assuming that the functions u, φ admit formal expansions with respect to ε relative to the equilibrium state $u = 0$, $\varphi = h_0$, namely

$$\begin{cases} \varphi = h_0 + \varepsilon \varphi' + \varepsilon^2 \varphi'' + \dots \, , \\ u = 0 + \varepsilon u' + \varepsilon^2 u'' + \dots \, . \end{cases} \tag{2.8}$$

([10]) C. H. Su and C. S. Gardner: *J. Math. Phys.*, **10**, 536 (1969).
([11]) G. B. Whitham: *Linear and Nonlinear Waves* (New York, N. Y., 1974).

No contradiction arises from the use of ε both in (2.7) and in (2.8) in that perturbing the solution relative to $u = 0$, $\varphi = h_0$ is in fact just the same as perturbing relative to the fundamental wave provided the amplitude of the fundamental wave be negligible against the quantity $\varepsilon\varphi'$. In other words, ε must be thought of as small but not infinitesimal.

In terms of ξ and τ the system (2.5) becomes

$$\begin{cases} \varepsilon\varphi_\tau + (u - c_0)\varphi_\xi + \varphi u_\xi = 0\,, \\ \varepsilon u_\tau + (u - c_0) u_\xi + \varphi^{-1}\mathscr{P}_\xi = 0\,. \end{cases} \tag{2.9}$$

On the other hand, consistently with (2.8) it is convenient to write $\mathscr{P}$ in the form

$$\mathscr{P} = \mathscr{P}_0 + \varepsilon\mathscr{P}' + \varepsilon^2\mathscr{P}'' + \dots\,.$$

Direct substitution of (2.8) and use of (2.7) yield

$$\mathscr{P}_0 = \tfrac{1}{2}c_0^2 h_0\,, \qquad \mathscr{P}' = c_0^2\varphi'\,, \qquad \mathscr{P}'' = c_0^2\varphi'' + \tfrac{1}{2}g\varphi'^2 + \tfrac{1}{3}c_0^2 h_0^2\varepsilon^{2\alpha-1}\varphi'_{\xi\xi}\,.$$

Hence, within the first-order approximation, eqs. (2.9) deliver

$$u'_\xi = \frac{c_0}{h_0}\varphi'_\xi\,,$$

whence

$$u' = \frac{c_0}{h_0}\varphi' + f(\tau)\,,$$

where f is an arbitrary function. At least two reasons contrast with the case $f \neq 0$. First, quite paradoxically $f \neq 0$ allows for the possibility $u' \neq 0$ albeit φ' vanishes identically and *vice versa*. Second, it is usually assumed that both u' and φ' decrease to zero as $\xi \to \pm\infty$. Then, if we let $f(\tau) \equiv 0$, in the second-order approximation, eqs. (2.9) give

$$\varphi'_\tau + 2\frac{c_0}{h_0}\varphi'\varphi'_\xi - (c_0\varphi''_\xi - h_0 u''_\xi) = 0\,,$$

$$\frac{c_0}{h_0}\varphi'_\tau + \frac{c_0^2}{h_0^2}\varphi'\varphi'_\xi + \frac{1}{3}c_0 h_0\varepsilon^{2\alpha-1}\varphi'_{\xi\xi\xi} + \frac{c_0}{h_0}(c_0\varphi''_\xi - h_0 u''_\xi) = 0\,.$$

By comparison it follows that

$$\varphi'_\tau + \frac{3}{2}\frac{c_0}{h_0}\varphi'\varphi'_\xi + \frac{1}{6}c_0 h_0^2\varepsilon^{2\alpha-1}\varphi'_{\xi\xi\xi} = 0\,,$$

which becomes the KdV equation as soon as we choose $\alpha = \frac{1}{2}$; in fact, setting $\varepsilon\varphi' = \eta$ and returning to the x, t co-ordinates, we find that

$$\eta_t + c_0\eta_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x + \frac{1}{6}c_0 h_0^2 \eta_{xxx} = 0\,. \tag{2.10}$$

At last, we may look for the outcome of the transformation

$$\xi = \varepsilon^{\alpha+1}x\,,$$

$$\tau = \varepsilon^{\alpha}\left(t - \frac{x}{c_0}\right),$$

instead of (2.7). In such a case we achieve a hardly interesting result. In fact, it is an easy matter to show that, following along the previous procedure, we find merely the equation arising from (2.10) by interchanging τ and ξ.

3. – Alternatives to the Korteweg-de Vries equation.

Recently some alternatives to the KdV equation have appeared in the literature [5,6]. The search for such alternatives is motivated on physical grounds. Indeed, a straightforward Fourier analysis of the linearized version of the KdV equation shows that the frequency ω goes as k^3 when the wave number k goes to infinity. This implies that both phase velocity $c_p = \omega/k$ and group velocity $c_g = d\omega/dk$ are unbounded as $k \to \infty$. In other words, the speed of propagation for the KdV equation is infinite [12]. Conversely, the exact solution for linearized waves in water of depth h_0 gives the dispersion relation [13]

$$\omega^2 = gk \operatorname{tgh} kh_0\,,$$

which shows that c_p and c_g decrease to zero as k goes to infinity.

In our opinion, it is just this observation which justifies a search for alternatives to the KdV equation subject to the requirement $c_p, c_g \to 0$ as $k \to \infty$. Moreover, the sought alternatives should satisfy further requirements such as to retain the same behaviour of the KdV equation when k is small enough, that is $\omega \simeq c_0 k - c_0 h_0^2 k^3/6$ and to preserve the sign of the velocities c_p, c_g. Here we outline whether, and how, the alternatives proposed in the literature satisfy the required conditions.

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(12) In connection with this point see, *e.g.*, T. LEVI-CIVITA: *Caratteristiche dei sistemi differenziali e propagazione ondosa* (Bologna, 1931) and appendix A of M. CARRASSI and A. MORRO: *Nuovo Cimento B*, **9**, 321 (1972).

(13) L. LANDAU and E. LIFCHITZ: *Mécanique des fluides* (Moscow, 1971).

The starting point for the derivation of the KdV equation is the fundamental wave satisfying $\eta_t = -c_0\eta_x$. On the basis of this observation, BENJAMIN, BONA and MAHONY suggested to replace the term η_{xxx} by $-\eta_{xxt}/c_0$; the resulting equation is

$$\eta_t + c_0\eta_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x - \frac{1}{6}h_0^2\eta_{xxt} = 0\,. \tag{3.1}$$

Both KdV and BBM equations involve only first-order derivatives with respect to time. Then, as remarked by JEFFREY, this implies the unpleasant feature that the specification of both η and η_t as initial data is not allowed. Furthermore, the linearized counterpart of (3.1) yields the dispersion relation

$$\omega = \frac{6c_0k}{6 + h_0^2k^2}\,.$$

First,

$$c_p = \frac{6c_0}{6 + h_0^2k^2}\,, \qquad c_g = 6c_0\frac{6 - h_0^2k^2}{(6 + h_0^2k^2)^2}$$

and hence $c_p, c_g \to 0$ as $k \to \infty$. Second,

$$\omega \simeq c_0k - \tfrac{1}{6}c_0h_0^2k^3$$

when k is small. Third, while c_p does not change its sign, c_g does.

Both to avoid the change of sign of c_g and to make it possible the assignment of η, η_t as initial data, JEFFREY proposed to use twice the approximation $\eta_t = -c_0\eta_x$, thus obtaining the J equation [14]

$$\eta_t + c_0\eta_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x + \frac{1}{6}\frac{h_0^2}{c_0}\eta_{xtt} = 0\,. \tag{3.2}$$

The linearized version of (3.2) yields

$$\omega = c_0\frac{(9 + 6h_0^2k^2)^{\frac{1}{2}} - 3}{h_0^2k}\,.$$

Hence,

$$c_p = c_0\frac{(9 + 6h_0^2k^2)^{\frac{1}{2}} - 3}{h_0^2k^2}\,, \qquad c_g = 3c_0\frac{(9 + 6h_0^2k^2)^{\frac{1}{2}} - 3}{h_0^2k^2(9 + 6h_0^2k^2)^{\frac{1}{2}}}\,,$$

whence it follows at once that c_p and c_g do not change sign while $c_p, c_g \to 0$

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[14] Called TRLW equation by JEFFREY.

as $k \to \infty$. Moreover,

$$\omega \simeq c_0 k - \tfrac{1}{6} c_0 h_0^2 k^3$$

when k is small.

At this stage the J equation could seem a model adequate to describe the propagation of small but finite-amplitude water waves. The following remarks aim to shed light on the physical significance of BBM and J equations. First, a direct deduction of (3.1) or (3.2), like that given for the KdV equation, should provide their immediate physical interpretation. Unfortunately, to our mind such deductions are not feasible. Indeed, the two-parameter transformation

$$\xi = \varepsilon^{\alpha}(x - c_0 t) ,$$
$$\tau = a\varepsilon^{\beta} t + b\varepsilon^{\gamma} x , \qquad \beta, \gamma \geqslant \alpha ,$$

applied to the system (2.5), allows reasonable solutions only if $b = 0$, thus leading to the KdV equation.

Second, as a matter of fact, the KdV equation (2.10) is Galileian invariant, that is to say invariant under the transformation ([15])

$$\text{(3.3)} \qquad \begin{cases} x \to x - Vt , \\ t \to t , \\ \eta \to \eta . \end{cases}$$

This is not the case for BBM and J equations.

As a final remark, we conjecture that the previous features of BBM and J equations are closely connected with one another. Indeed, (2.7) privileges the quantity $x - c_0 t$ and this is meaningful only if the resulting equation is Galileian invariant. Furthermore, the nonderivation of BBM and J equations from GN equations is hardly surprising, since the GN model relies heavily on the Galileian invariance ([1,7]).

4. – Preliminary topics about variational principles in hydrodynamics.

The search for variational principles in hydrodynamics is essentially suggested by a twofold argument. First, as happens in other fields, the existence of a variational principle, besides resulting in a unification of the subject,

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([15]) Galileian invariance is considered also by R. M. MIURA: *SIAM Rev.*, **18**, 412 (1976) through the transformation $x \to x - Vt$, $t \to t$, $\eta \to \eta + 2h_0 V/3c_0$, which seems to be suggested by formal reasons. On the contrary, (3.3) is motivated by the fact that the physical meaning of the scalar quantity η is preserved only if its value is the same in all Galileian frames.

may lead to new methods for solving the problem. Second, in connection with nonlinear wave theories it is desirable to have a technique for distinguishing between those wave equations that allow for dissipation and those that do not. On the other hand, it is customary to consider conservative (or nondissipative) those wave systems which admit a Lagrangian density, even if not in the sense of classical mechanics, thus motivating the afore-mentioned search.

Unfortunately, we are still short of a general routine method for producing variational principles. The easiest way to see how special the known variational formulations are is to examine them as we do in the next section. Here, instead, we outline an alternative deduction of standard equations for water waves by emphasizing the role played by the free boundary of the fluid in the variational formulation.

On assuming the irrotationality of the velocity field and denoting by $\Phi(\boldsymbol{x}, t)$ the velocity potential, that is $\boldsymbol{V}(\boldsymbol{x}, t) = \nabla\Phi + \Phi_z \boldsymbol{e}_3$, a variational principle for a fluid with a free surface may be written in the form [11,16]

$$(4.1)\qquad \delta J(\Phi) = 0\,, \qquad J(\Phi) = \int_R \int_{-h}^{\eta} \left(\Phi_t + \tfrac{1}{2}(\nabla\Phi)^2 + \tfrac{1}{2}\Phi_z^2 + gz\right) \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t\,,$$

where $\eta = \eta(x, y, t)$, $h = h(x, y)$ and R is the cylindrical region $D\times[t_1, t_2]$. Borrowing from Hamilton's principle in classical mechanics the unknown function Φ is assumed to be fixed at times t_1, t_2. On the other hand, owing to the presence of a free boundary at $z = \eta$, η too is an unknown function for the problem at hand. Thus we are led to consider the differential δJ arising from a change of Φ, namely

$$\Phi \to \Phi + \chi\,, \qquad \chi(t_1) = \chi(t_2) = 0\,,$$

and a change of the domain, that is

$$\boldsymbol{x} \to \boldsymbol{x} + \alpha \boldsymbol{e}_3\,, \qquad \alpha = 0 \ \text{at} \ z = -h\,.$$

As a consequence δJ takes the form [17]

$$\delta J = -\int_R \int_{-h}^{\eta} (\nabla^2\Phi + \Phi_{zz})\chi\, \mathrm{d}V + \int_{\partial\Omega} \left(\Phi_t + \tfrac{1}{2}(\nabla\Phi)^2 + \tfrac{1}{2}\Phi_z^2 + gz\right)\alpha \boldsymbol{e}_3\cdot\boldsymbol{n}\, \mathrm{d}\sigma + \\ + \int_{\partial\Omega} \{(\nabla\Phi + \Phi_z \boldsymbol{e}_3)\cdot\boldsymbol{n} + \nu\}\chi\, \mathrm{d}\sigma\,,$$

([16]) J. C. LUKE: *J. Fluid Mech.*, **27**, 395 (1967).
([17]) I. M. GELFAND and S. V. FOMIN: *Calculus of Variations*, subsect. **37**.4 (Englewood Cliffs, N. J., 1963).

where $\partial\Omega$ is the smooth boundary of $\Omega = R\times[-h, \eta]$ and $(\boldsymbol{n}, \nu)$ is the outward unit normal to $\partial\Omega$. First, if $\chi = 0$ and $\alpha = 0$ at $\partial\Omega$, $\delta J = 0$ yields the Euler-Lagrange equation

$$\nabla^2\Phi + \Phi_{zz} = 0 \,. \tag{4.2}$$

Second, the choice $\chi = 0$ at $\partial\Omega$ gives

$$\Phi_t + \tfrac{1}{2}(\nabla\Phi)^2 + \tfrac{1}{2}\Phi_z^2 + gz = 0 \qquad \text{at } z = \eta \,. \tag{4.3}$$

Then we have

$$\delta J = \int\limits_{\partial\Omega} \{(\nabla\Phi + \Phi_z \boldsymbol{e}_3)\cdot\boldsymbol{n} + \nu\}\,\chi\, \mathrm{d}\sigma \,.$$

Observe that χ is required to vanish at the surfaces $t = t_1$, $t = t_2$; thus it is worth introducing the subset Σ obtained from $\partial\Omega$ by removing such surfaces. If we now make use of the arbitrariness of χ at Σ, the condition $\delta J = 0$ provides

$$(\nabla\Phi + \Phi_z \boldsymbol{e}_3)\cdot\boldsymbol{n} + \nu = 0 \qquad \text{at } \Sigma \,. \tag{4.4}$$

In particular, at $z = \eta$ we have

$$\boldsymbol{n} = \lambda(-\nabla\eta + \boldsymbol{e}_3) \,, \qquad \nu = -\lambda\eta_t \,, \qquad \lambda = \{(\nabla\eta)^2 + \eta_t^2 + 1\}^{-\frac{1}{2}} \,,$$

and hence (4.4) reduces to

$$\Phi_z - \eta_t - \nabla\eta\cdot\nabla\Phi = 0 \qquad \text{at } z = \eta \,. \tag{4.5}$$

Analogously, at $z = -h$ we have $\boldsymbol{n} = \mu(\nabla h + \boldsymbol{e}_3)$, $\nu = 0$, $\mu = \{(\nabla h)^2 + 1\}^{-\frac{1}{2}}$, and then (4.4) simplifies to

$$\Phi_z + \nabla h\cdot\nabla\Phi = 0 \qquad \text{at } z = -h \,. \tag{4.6}$$

To sum up, we have seen that the variational principle (4.1) leads to the well-known equation of motion (4.2) and boundary conditions (4.3), (4.5), (4.6) for the water wave problem.

In so doing we have not exploited the boundary condition (4.4) when referred to the subset $\Sigma^\dagger$ obtained from Σ by deleting the surfaces $z = \eta$, $z = -h$. This is because the fluid is usually considered to have infinite extension with respect to the co-ordinates x, y. If such is not the case, in following along the above procedure, (4.4) delivers the right boundary condition at $\Sigma^\dagger$.

As is well known, the full set of equations of motion is unwieldy in practical problems. This in turn confines the interest of the variational formulation (4.1) to theoretical frameworks. On the other hand, several approximate theories for describing the fluid motion are now available. Thus the problem is to find, if possible, their variational counterparts. This subject is examined in the next sections.

5. – Some examples of variational principles.

This section deals with the variational principles corresponding to the models outlined in sect. **2**, **3**. As they stand, some systems of differential equations seem not to follow from variational principles. Yet, as happens also in other contexts, a nontrivial change of unknown functions makes the variational formulation immediate. Note that such *ad hoc* procedures stress once again the lack of systematic methods of finding variational principles in hydrodynamics.

Shallow-water theory. Consider the Lagrangian density

$$\mathscr{L} = (\eta + h)\big(\Phi_t + \tfrac{1}{2}(\nabla\Phi)^2\big) + \tfrac{1}{2}g\eta^2 ,$$

where the unknown functions are the profile η and the velocity potential Φ. The corresponding Euler-Lagrange equations are

$$\begin{aligned} \eta: &\quad \Phi_t + \tfrac{1}{2}(\nabla\Phi)^2 + g\eta = 0 , \\ \Phi: &\quad \eta_t + \nabla\cdot\{(\eta + h)\,\nabla\Phi\} = 0 . \end{aligned}$$

Boussinesq equation. From the Lagrangian density

$$\mathscr{L} = (\eta + h)\big(\Phi_t + \tfrac{1}{2}(\nabla\Phi)^2\big) + \tfrac{1}{2}g\eta^2 - \tfrac{1}{6}h_0\eta_t^2$$

we get the Euler-Lagrange equations

$$\begin{aligned} \eta: &\quad \Phi_t + \tfrac{1}{2}(\nabla\Phi)^2 + g\eta + \tfrac{1}{3}h_0\eta_{tt} = 0 , \\ \Phi: &\quad \eta_t + \nabla\cdot\{(\eta + h_0)\,\nabla\Phi\} = 0 . \end{aligned}$$

Korteweg-de Vries equation. If we let $\theta_x = 3\eta/2h_0$, the Lagrangian density

$$\mathscr{L} = \frac{1}{2}\theta_x\theta_t + \frac{1}{2}c_0\theta_x^2 + \frac{1}{6}c_0\theta_x^3 + \frac{1}{12}c_0h_0^2(\chi^2 + 2\chi_x\theta_x)$$

in the unknown functions θ, χ allows us to write the Euler-Lagrange equations

$$\begin{aligned} \chi: &\quad \chi - \theta_{xx} = 0 , \\ \theta: &\quad \theta_{xt} + c_0\theta_{xx} + c_0\theta_x\theta_{xx} + \tfrac{1}{6}c_0h_0^2\chi_{xx} = 0 , \end{aligned}$$

which are equivalent to KdV equation.

Jeffrey equation. If we let $\theta_x = 3\eta/2h_0$, it follows at once that the Euler-Lagrange equations corresponding to

$$\mathscr{L} = \frac{1}{2}\theta_x\theta_t + \frac{1}{2}c_0\theta_x^2 + \frac{1}{6}c_0\theta_x^3 + \frac{1}{12}\frac{h_0^2}{c_0}(\chi + 2\chi_x\theta_t)$$

are

$$\chi: \qquad \chi - \theta_{xt} = 0\,,$$

$$\theta: \qquad \theta_{xt} + c_0\theta_x + c_0\theta_x\theta_{xx} + \frac{1}{6}\frac{h_0^2}{c_0}\chi_{xt} = 0\,.$$

The variational principle for J equation is so established.

The above principles are referred to irrotational motions; nonzero vorticity motions are considered in the next section in conjunction with the GN model.

6. – Lin-like variational principle for Green and Naghdi's equations.

When looking for variational formulations in continuum physics we have preliminarly to decide whether the motion is described through Lagrangian or Eulerian co-ordinates. In fluid dynamics the Eulerian description appears to be preferable, though further difficulties arise out just in connection with variational problems. In fact, the crucial difference is that in the Lagrangian description the motion is specified through the position of the particles, whereas in the Eulerian description it is represented by the velocity field. Thus, while a natural analogy with particle mechanics is profitable in the Lagrangian description, such an analogy is not available in the Eulerian description. To overcome this difficulty LIN [18] pointed out that a particlelike information on the motion may be introduced through the obvious constraint

$$\dot{\alpha} \equiv \frac{\partial \alpha}{\partial t} + v_j\frac{\partial \alpha}{\partial x_j} = 0\,,$$

$\alpha(\boldsymbol{x}, t)$ being the Lagrangian label of the particle in $\boldsymbol{x}$ at the time t. Accounting also for the constraint related to the mass conservation, Lin's procedure leads to the equations of motion as the Clebsch equations [19]. Such a procedure is here applied in connection with the GN model.

Roughly speaking, the column structure of the GN model allows us to imagine the fluid motion as resulting from the horizontal motion of the columns and the vertical motion inside the columns themselves. Accordingly, it is

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[18] C. C. LIN: *Rendiconti S.I.F.*, Course XXI (New York, N. Y., 1963).

[19] H. LAMB: *Hydrodynamics*, VI ed. (Cambridge, 1932), art. 166.

convenient to consider an equivalent bidimensional layer obtained by shrinking the fluid vertically. This is made mathematically precise in the following way. Let $\varkappa = \varrho\varphi$ be the density of the layer described by the equation

$$\varkappa_t + \nabla\cdot(\varkappa\boldsymbol{v}) = 0\,,$$

$$\varkappa\boldsymbol{v}_t + \varkappa(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla\Pi + p_a\nabla\eta + P\,\nabla h\,,$$

which are an immediate consequence of $(2.1)_{1,2}$. On using the obvious identity

$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \nabla(\tfrac{1}{2}\boldsymbol{v}^2) + (\nabla\wedge\boldsymbol{v})\wedge\boldsymbol{v}$$

and appealing to the Clebsch potential in the form

$$\boldsymbol{v} = \nabla\chi + \mu\,\nabla\sigma\,,$$

we find that

$$\boldsymbol{v}_t + (\nabla\wedge\boldsymbol{v})\wedge\boldsymbol{v} = \nabla(\chi_t + \mu\sigma_t) + \mu\,\nabla\sigma - \sigma\,\nabla\mu\,.$$

Under the assumption $\dot{\mu} = 0$, $\dot{\sigma} = 0$, we can write

$$\varkappa(\chi_t + \mu\sigma_t + \tfrac{1}{2}\boldsymbol{v}^2) = \boldsymbol{p}\,,$$

where $\boldsymbol{p} = -\nabla\Pi + p_a\nabla\eta + P\,\nabla h$. A direct integration yields

$$\chi_t + \mu\sigma_t + \frac{1}{2}\boldsymbol{v}^2 = \int\frac{\boldsymbol{p}\cdot\mathrm{d}\boldsymbol{r}}{\varkappa}\,.$$

The previous scheme is unified by the following variational principle:

$$\delta\int_{t_1}^{t_2}\int_D\left(\tfrac{1}{2}\varkappa\boldsymbol{v}^2 - E + \chi\{\varkappa_t + \nabla\cdot(\varkappa\boldsymbol{v})\} + \sigma\{(\varkappa\mu)_t + \nabla\cdot(\varkappa\mu\boldsymbol{v})\}\right)\mathrm{d}\boldsymbol{r}\,\mathrm{d}t = 0\,,$$

E being defined by

$$E_\varkappa = -\int\frac{\boldsymbol{p}\cdot\mathrm{d}\boldsymbol{r}}{\varkappa}\,.$$

Indeed, the Euler-Lagrange equations are

$$\begin{aligned}
\boldsymbol{v}&: \quad \boldsymbol{v} = \nabla\chi + \mu\,\nabla\sigma\,,\\
\varkappa&: \quad \tfrac{1}{2}\boldsymbol{v}^2 - E_\varkappa = \chi_t + \boldsymbol{v}\cdot\nabla\chi + \mu\dot{\sigma}\,,\\
\chi&: \quad \varkappa_t + \nabla\cdot(\varkappa\boldsymbol{v}) = 0\,,\\
\mu&: \quad \dot{\sigma} = 0\,,\\
\sigma&: \quad \dot{\mu} = 0\,.
\end{aligned}$$

They reduce to the corresponding ones given above through straightforward algebra.

The motion of the layer is so completely characterized. Of course, to determine the actual motion of the fluid, use must be done of eqs. $(2.1)_{1,2}$.

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RIASSUNTO

Si presenta una trattazione compatta delle più note equazioni che descrivono la propagazione delle onde d'acqua. Un primo schema riunisce le equazioni dell'acqua bassa, di Boussinesq e di Korteweg-de Vries come casi particolari delle equazioni di Green e Naghdi. Per contro, l'equazione di Benjamin-Bona-Mahony e quella di Jeffrey non rientrano in tale schema: ciò è attribuito al fatto che esse non sono invarianti per trasformazioni di Galileo. Un secondo schema mette in rilievo le diverse caratteristiche dei vari modelli tramite le corrispondenti formulazioni variazionali. In questo contesto si fornisce un principio variazionale per le equazioni di Green e Naghdi.

Теория водяных волн и вариационные принципы.

Резюме (*). — Предлагается единое описание моделей для распространения водяных волн. Первая схема описывает уравнения мелкой воды, уравнения Буссинэ и уравнение Кортевега-де Вриса, как частные случаи уравнений Грина и Нагди. Уравнения Бенджамина-Бона-Махони и Джефри не могут быть получены в такой схеме: это приписывается негалилеевой инвариантности этих уравнений. Вторая схема придает особое значение особенностям различных моделей посредством соответствующих вариационных формулировок. В этом контексте устанавливается вариационный принцип для уравнений Грина и Нагди.

(*) *Переведено редакцией.*