IL NUOVO CIMENTO VOL. 1 C, N. 5 Settembre-Ottobre 1978

Gravity Waves in Water of Variable Depth.

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(ricevuto il 3 Agosto 1978)

Summary. — This work is concerned with the analysis of some models for gravity wave propagation in water of variable depth. Within this framework, particular emphasis is put on the model recently proposed by Green and Naghdi. A detailed comparison with widely accepted theories allows us to consider the model by Green and Naghdi essentially as a theory closely related to the approximation of shallow water. However, the greater generality of this model, outlined in the present paper, opens new prospects of the possibility of achieving more satisfactory results about gravity wave propagation.

1. – Introduction.

The study of gravity wave (water wave) propagation reveals difficulties originated by the occurrence of the nonlinear inertia terms and of the nonlinear boundary condition over an unknown surface. The existence of these difficulties, together with the fact that the more interesting aspects of gravity wave propagation have two-dimensional character, motivates the development of some methods in order to replace the three-dimensional theory of gravity waves by approximate two-dimensional theories. In this connection, we recall that well-known methods of approximation are essentially to introduce one or more adimensional parameters which, in some sense, are regarded as small. In this way the so-called asymptotic expansions are obtained and these in turn lead to equations well known in the current literature. For a brief review of these procedures we cite, for instance, the papers by FRIEDRICHS (1), KELLER (2),

(1) K. O. FRIEDRICHS: *Comm. Pure Appl. Math.*, **1**, 109 (1948).
(2) J. B. KELLER: *Journ. Fluid Mech.*, **4**, 607 (1958).

PEREGRINE ([3,4]), GRIMSHAW ([5]) and JOHNSON ([6]). Although these methods of approximation appear as useful tools in applications, they suffer from a lack in proof of the asymptotic character of the expansion considered. Moreover, these methods require *a priori* assumptions about the scaling of the variables under study. Finally, again with respect to methods of approximation, we mention bidimensional theories based on the employ of conformal transformations which make the bottom horizontal ([7,8]).

With a view to overcoming the incomplete nature of the methods quoted above, in a recent paper ([9]) GREEN and NAGHDI attempted to derive, in a systematic way, the equation of motion for inviscid, nonconducting, homogeneous and incompressible fluids within the framework of a three-dimensional theory. In synthesis, GREEN and NAGHDI start with the usual integral version of the energy balance equation for an incompressible fluid. Then, by assuming the invariance of the energy equation under superposed rigid-body translation, they deduce the corresponding equations of motion together with the continuity equation. The particular feature that distinguishes the model by GREEN and NAGHDI (GN model) from normally adopted theories relies on the assumption that the vertical component of the fluid velocity depends linearly on and the horizontal component is independent of the vertical co-ordinate. In terms more suggestive than precise, this assumption can be phrased by saying that, according to the GN model, the elementary constituents of the fluid are infinitesimal vertical columns rather than the usual fluid particles. In other words, the fluid particles which, at some initial time, belong to a vertical (material) column will *continue* to belong to the same vertical column. Thus the horizontal velocity of a particle coincides with the horizontal velocity of the column. This description of the motion is indeed particularly suited for a coincise account of the boundary condition (at the bed and at the *unknown* free surface) and of the incompressibility.

The purpose of the present paper is twofold. First, we analyse the GN model within the framework of the current literature regarding wave propagation in fluid. To this end, in sect. **2**, we briefly review the work by GREEN and NAGHDI, while, in sect. **3**, we introduce the well-known shallow-water theory and small-amplitude theory. The subsequent comparison with these theories enables us to assert that the GN model, although more general, can be framed within the context of shallow-water theories. Second, we outline the precise limitations to the actual range of applicability of the GN model. To do this, it is sufficient

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([3]) D. H. PEREGRINE: *Journ. Fluid Mech.*, **27**, 815 (1967).
([4]) D. H. PEREGRINE: in *Waves on Beaches* (London, 1972), p. 95.
([5]) R. GRIMSHAW: *Journ. Fluid Mech.*, **42**, 639 (1970).
([6]) R. S. JOHNSON: *Proc. Camb. Phil. Soc.*, **73**, 183 (1973).
([7]) G. KEISEL: *Quart. Appl. Math.*, **7**, 21 (1949).
([8]) J. HAMILTON: *Journ. Fluid Mech.*, **83**, 289 (1977).
([9]) A. E. GREEN and P. M. NAGHDI: *Journ. Fluid Mech.*, **78**, 237 (1976).

to consider the simple physical case of wave propagation in water of constant depth (sect. 4). This study shows up an interesting feature of the theory, namely a dispersion relation whose main consequences are the following. First of all, at low frequencies we find the usual result for the velocity of propagation. Moreover, the dispersion relation reveals the existence of a cut-off frequency ω_c in the sense that the possibility of wave propagation is ruled out for frequencies greater than ω_c. Since ω_c decreases with the depth of the fluid, this conclusion allows us to infer that the column model is physically realistic when the water is shallow or the wave is long. We conclude sect. 4 by pointing out the possible applications of the theory by GREEN and NAGHDI, namely propagation of small-amplitude waves in water of variable depth.

2. – The model of Green and Naghdi.

As far as possible, in the following we use the original notations of GREEN and NAGHDI [9]. Accordingly, x, y, z denote the usual Cartesian orthogonal co-ordinates whose unit vectors are $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, respectively. The symbols ξ_t, ξ_x, ξ_y stand for the partial derivative of any quantity ξ with respect to the time t and the co-ordinates x, y. A superimposed dot indicates the material time derivative.

Let us consider an inviscid, homogeneous, incompressible fluid with constant mass density ϱ moving over an uneven bottom specified by

$$\boldsymbol{x} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 - h(x,y)\boldsymbol{e}_3 . \tag{2.1}$$

The free surface of the fluid is described by

$$\boldsymbol{x} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + \zeta(x, y, t)\boldsymbol{e}_3 . \tag{2.2}$$

Obviously, $h(x, y)$ is an *a priori* given function, whereas the function $\zeta(x, y, t)$ is unknown for the problem under study. We choose the origin of the vertical axis so that $z = 0$ is the free surface at the equilibrium (see fig. 1). By neglecting

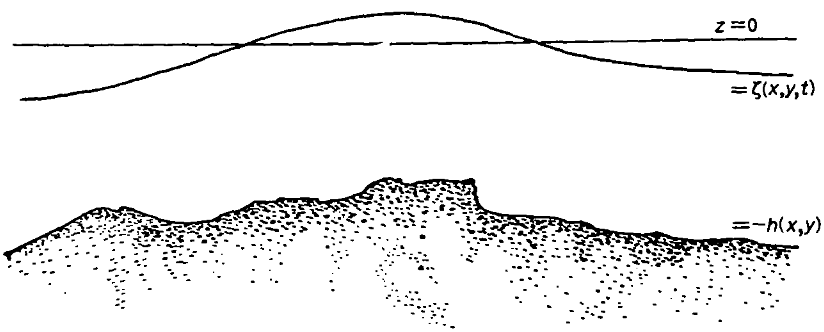

Fig. 1. – Geometry of the general propagation problem.

surface tension effects, the pressure at the free surface $z = \zeta(x, y, t)$ coincides with the atmospheric pressure p_a. Owing to the motion of the fluid, the unknown pressure P at the bed depends on x, y and also on t.

Let X_1, X_2, X_3 be the (Lagrangian) co-ordinates of a fluid particles with respect to a suitable reference configuration. The GN model relies on the assumption that the horizontal component of the velocity be independent of X_3 and the vertical component be a linear function of X_3. This means that a vertical column of fluid around (X_1, X_2) in the reference configuration remains vertical column around $(x(X_1, X_2, t), y(X_1, X_2, t))$ in the present configuration, while the vertical velocity component varies linearly along the column itself. This viewpoint makes it natural to express the position of a particle of the fluid in the form

$$\boldsymbol{x} = \boldsymbol{r} + (\psi + X\varphi)\boldsymbol{e}_3 , \tag{2.3}$$

where $\boldsymbol{r} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2$, $\psi = (\zeta - h)/2$, $\varphi = \zeta + h$, $X \equiv X_3$. It follows at once that the free-fluid surface (2.2) may be obtained by setting $X = \frac{1}{2}$ in (2.3), while the bed (2.1) corresponds to $X = -\frac{1}{2}$; consequently $X \in [-\frac{1}{2}, \frac{1}{2}]$. In view of (2.3) the velocity $\boldsymbol{V} = \dot{\boldsymbol{x}}$ may be written in the form

$$\boldsymbol{V} = \boldsymbol{v} + (\lambda + Xw)\boldsymbol{e}_3 , \tag{2.4}$$

$\boldsymbol{v} = \dot{\boldsymbol{r}} = u\boldsymbol{e}_1 + v\boldsymbol{e}_2$ being the horizontal component of the velocity and $\lambda = \dot{\psi}$, $w = \dot{\varphi}$. As to the physical meaning of λ and w, we note that $\lambda(x, y, t)$ is the vertical velocity of the centre of mass of the fluid column around (x, y), while Xw is the vertical velocity of the particles within the column, relative to the centre of mass.

Let us consider now an arbitrary fluid column occupying a region $\mathscr{P}^*$ bounded by a closed cylinder $\partial\mathscr{P}^*_n$, whose unit outwards normal is denoted by $\boldsymbol{n}$, defined by an equation of the form $f(X_1, X_2) = 0$ (see fig. 2).

Letting $\mathscr{P}$ stand for the part of the surface $z = \psi(x, y, t)$ belonging to $\mathscr{P}^*$, we can write

$$\int_{\partial\mathscr{P}^*_n} p\boldsymbol{V}\cdot\boldsymbol{n}\,\mathrm{d}a = \int_{\partial\mathscr{P}^*_n} p\boldsymbol{e}\cdot\boldsymbol{v}\wedge\mathrm{d}\boldsymbol{r}\,\mathrm{d}z = \int_{\partial\mathscr{P}} \Pi(\boldsymbol{e}_3\wedge\boldsymbol{v})\cdot\mathrm{d}\boldsymbol{r} , \tag{2.5}$$

where $p(x, y, z, t)$ is the pressure and Π is defined by

$$\Pi = \int_{-h}^{\zeta} p\,\mathrm{d}z .$$

The force acting on an infinitesimal surface element $\boldsymbol{n}\,\mathrm{d}a$ of the fluid is given by

$$-p\boldsymbol{n}\,\mathrm{d}a = p_a(\zeta_x\boldsymbol{e}_1 + \zeta_y\boldsymbol{e}_2 - \boldsymbol{e}_3)\,\mathrm{d}x\,\mathrm{d}y , \qquad z = \zeta(x, y, t) \tag{2.6}$$

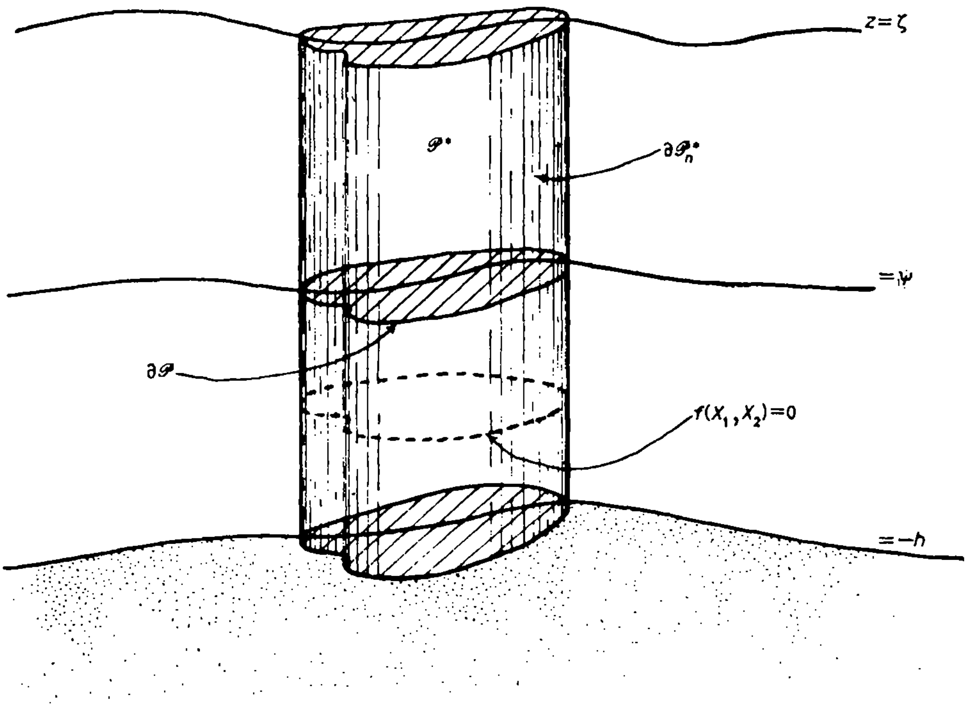

Fig. 2. - Fluid column $\mathscr{P}^*$.

at the free surface and by

$$-p\boldsymbol{n}\,\mathrm{d}a = P(h_x\boldsymbol{e}_1 + h_y\boldsymbol{e}_2 - \boldsymbol{e}_3)\,\mathrm{d}x\,\mathrm{d}y\,, \qquad z = -h(x,y) \tag{2.7}$$

at the bottom. The use of eqs. (2.5)-(2.7) enables the energy balance equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*}\varrho\left(\frac{1}{2}V^2 + gz\right)\mathrm{d}v = -\int_{\partial\mathscr{P}^*} p\boldsymbol{V}\cdot\boldsymbol{n}\,\mathrm{d}a \tag{2.8}$$

to be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\frac{1}{2}\varrho\varphi\left(v^2 + \frac{1}{12}w^2 + 2g\psi\right)\mathrm{d}x\,\mathrm{d}y = \tag{2.9}$$

$$= \int_{\mathscr{P}}\left\{p_*\left(-\boldsymbol{v}\cdot\nabla\zeta + \lambda + \frac{1}{2}w\right) - P\left(\boldsymbol{v}\cdot\nabla h + \lambda - \frac{1}{2}w\right)\mathrm{d}x\,\mathrm{d}y - \boldsymbol{e}_3\cdot\int_{\partial\mathscr{P}}\Pi\boldsymbol{v}\wedge\mathrm{d}\boldsymbol{r}\right..$$

The symbol $\nabla := \boldsymbol{e}_1\partial/\partial x + \boldsymbol{e}_2\partial/\partial y$ denotes the bidimensional gradient operator.

The equations written above are not sufficient for a complete theory since, so far, we have not considered the equations of motion of the fluid. To this purpose, we consider the energy balance in the form (2.9) and we assume that it is invariant under superposed rigid-body motion. This means that the change of frame corresponding to the trasformation $\boldsymbol{V} \to \boldsymbol{V} + \boldsymbol{U}$ leaves (2.9) unaltered. Then the arbitrariness of the constant vector $\boldsymbol{U}$ allows us to write the relations

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \varrho\varphi \,\mathrm{d}x\,\mathrm{d}y = 0\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \varrho\varphi\boldsymbol{v} \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathscr{P}} (p_a \nabla\zeta + P\nabla h)\,\mathrm{d}x\,\mathrm{d}y + \mathbf{e}_3 \wedge \int_{\partial\mathscr{P}} \Pi \,\mathrm{d}\mathbf{r}\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \varrho\varphi(\lambda + gt)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathscr{P}} (P - p_a)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Under suitable smoothness assumptions for the fields under study, these equations provide the corresponding local field equations

$$\dot{\varphi} + \varphi\nabla\cdot\boldsymbol{v} = 0\,, \tag{2.10}$$

$$\varrho\varphi\dot{\boldsymbol{v}} = -\nabla\Pi + p_a\nabla\zeta + P\nabla h\,, \tag{2.11}$$

$$\varrho\varphi\dot{\lambda} = P - p_a - \varrho g\varphi\,. \tag{2.12}$$

Moreover, on account of (2.10)-(2.12) the energy balance (2.9) yields

$$\tfrac{1}{12}\varrho\varphi^2\dot{w} = \Pi - \tfrac{1}{2}(P + p_a)\varphi\,. \tag{2.13}$$

For later purposes, it is worth remarking that, according to (2.12), the vertical acceleration of the centre of mass $\dot{\lambda}$ is singled out by the difference between dynamic and hydrostatic pressure at the bottom. This feature will be returned to in sect. **3** in connection with the shallow-water approximation.

The set of equations (2.10)-(2.13) constitutes a system of five scalar (nonlinear) equations in the five unknown functions ζ (or $\varphi = \zeta + h$), $\boldsymbol{v}$, P, Π. Even because of the nonlinearity, in general it is a hard task to determine the motion of the fluid according to the GN model described above. On the other hand, a comparison between this model and those usually adopted in the literature is not immediate. This motivates a detailed analysis of the GN model in connection with the more familiar models such as the shallow-water theory and the small-amplitude theory. We are dealing with this argument in the next section.

3. – Some models for water wave propagation.

The key problem we are concerned with is the description of the wave motion of a fluid. For definiteness, we say that, in general, a *wave motion* of a fluid, acted upon by gravity and having a free surface, is a motion in which the quantity $\zeta(x, y, t)$ varies in time (ref. ([10]), subsect. **14**.30). More specific wave motions are widely considered by several authors; for the reader's convenience, we list here the most common types of wave motions.

A wave motion is called a *standing wave* when the fluid surface changes its shape by moving vertically without translation. A typical standing wave is expressed by (ref. ([11]), subsect. **3**.1)

$$\zeta(x, y, t) = \mathrm{Re}\,\{z(x, y)\exp[i\omega t]\}\,. \tag{3.1}$$

A wave motion is called a *progressive wave* when there exists a frame of reference for which the motion is time independent ([12]). A particular progressive wave is the *simple harmonic progressive wave* given by

$$\zeta(x, y, t) = \mathrm{Re}\,\{a\exp[i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)]\}\,, \tag{3.2}$$

where a, $\boldsymbol{k}$, ω are constant quantities. In connection with a simple harmonic progressive wave propagating over a basin of depth h, it is worth introducing two adimensional parameters, namely ([3])

$$\varepsilon = \frac{a}{h}\,,\quad \sigma = hk \qquad (k = |\boldsymbol{k}|)\,. \tag{3.3}$$

In the limiting case $h = \infty$ we define $\varepsilon\sigma = ak$.

With this in mind, we would like to examine the standard models a little more closely in regard to the GN model.

i) *Shallow-water theory* (long-wave theory). The shallow-water theory is based on the assumption that the vertical component of the water particle acceleration has a negligible effect on the pressure p (ref. ([11]), subsect. **2**.2). Mathematically, this is tantamount to assuming that the dynamic pressure p is approximately equal to the hydrostatic pressure, *i.e.*

$$p \simeq \varrho g(\zeta - z) + p_a\,. \tag{3.4}$$

([10]) L. M. Milne and C. B. E. Thomson: *Theoretical Hydrodynamics* (London, 1968).
([11]) J. J. Stoker: *Water Waves* (New York, N. Y., 1957).
([12]) T. Levi Civita: *Math. Ann.*, **93**, 264 (1925).

This model is also commonly referred to as the theory of long wave since the relation (3.4) (and (3.7)) can be deduced by means of a perturbation procedure involving a formal development of all quantities in powers of the small parameter σ (ref. [13], sect. **171**; [14]). However, as noted frankly by STOKER in ref. [11], p. 31, the quoted method does not *prove* that (3.4) is, in some sense, an appropriate assumption as the formal developments are introduced in just such a way that (3.4) would result [15].

With respect to the GN model, on account of the definitions of P and Π, the assumption (3.4) implies

$$\begin{cases} P \simeq \varrho g(\zeta + h) + p_a\,, \\ \Pi \simeq (\frac{1}{2}\varrho g(\zeta + h) + p_a)(\zeta + h)\,. \end{cases} \tag{3.5}$$

Substitution of the relations (3.5) into (2.12), (2.13) yields

$$\dot{\lambda} \simeq 0\,, \qquad \dot{w} \simeq 0\,. \tag{3.6}$$

In view of definition (2.4), these results in turn lead to the starting point according to which the vertical component of the acceleration is negligible. In the meantime eqs. (2.10), (2.11) provide the fundamental equations of the *nonlinear shallow-water theory* (finite-amplitude long-wave theory)

$$\begin{cases} \zeta_t + \nabla\cdot[(\zeta + h)\boldsymbol{v}] = 0\,, \\ \boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\,g\nabla\zeta\,, \end{cases} \tag{3.7}$$

in complete agreement with the analogous ones presented in the literature (see, *e.g.*, ref. [11], subsect. **2**'2). The standard linearization procedure reduces the system (3.7) to the well-known *linearized shallow-water theory* differential equations

$$\begin{cases} \zeta_t + \nabla\cdot(h\boldsymbol{v}) = 0\,, \\ \boldsymbol{v}_t = -\,g\nabla\zeta\,. \end{cases} \tag{3.8}$$

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[13] H. LAMB: *Hydrodynamics*, VI ed. (Cambridge, 1932).
[14] Further details about these perturbation procedures may be found, *e.g.*, in ref. [3], [11], subsect. **2**'4; see also K. O. FRIEDRICHS: *Comm. Pure Appl. Math.*, **1**, 81 (1948); J. B. KELLER: *Comm. Pure Appl. Math.*, **1**, 323 (1948).
[15] Nevertheless, it is worth remarking that K. O. FRIEDRICHS and D. H. HYERS: *Comm. Pure Appl. Math.*, **7**, 517 (1954), have shown the development does yield the existence of the solitary-wave solution.

The classical linear hyperbolic equation (see, *e.g.*, ref. [13], sect. **191**; [16]; [17] sect. **13**)

$$\zeta_{tt} - g\nabla\cdot[h\nabla\zeta] = 0 \tag{3.9}$$

follows immediately.

ii) *Small-amplitude theory* (linear approximation). This model hinges on the assumption that the variables under consideration slightly differ from the values at hydrostatic equilibrium (still water). This is equivalent to the requirement $\varepsilon \ll 1$ [18]. This model can be derived as an approximation to the general theory by assuming that all the variables possess a suitable power series expansion with respect to ε (ref. [11], subsect. **2·1**). Otherwise we can proceed as follows. The pressure at the bed P may be written in the form

$$P = p_a + \varrho g h + \mathfrak{p}\,;$$

by definition $\mathfrak{p}$ is the difference between the dynamic and the hydrostatic pressure. Analogously, we set

$$\Pi = p_a h + \tfrac{1}{2}\varrho g h^2 + \pi\,.$$

By neglecting nonlinear terms with respect to the perturbations ζ, $\boldsymbol{v}$, $\mathfrak{p}$, π, we obtain the differential equations of the *small-amplitude wave theory* within the framework of the GN model

$$\left|\begin{aligned} &\zeta_t + \nabla\cdot(h\boldsymbol{v}) = 0\,,\\ &\varrho h \boldsymbol{v}_t = -\nabla\pi + p_a\nabla\zeta + \mathfrak{p}\nabla h\,,\\ &\tfrac{1}{2}\varrho h(\zeta_{tt} - \boldsymbol{v}_t\cdot\nabla h) = -\varrho g\zeta + \mathfrak{p}\,,\\ &\tfrac{1}{12}\varrho h^2(\zeta_{tt} + \boldsymbol{v}_t\cdot\nabla h) = \pi - \tfrac{1}{2}h\mathfrak{p} - (p_a + \tfrac{1}{2}\varrho g h)\zeta\,,\end{aligned}\right. \tag{3.10}$$

where $\lambda_t \simeq \frac{1}{2}(\zeta_{tt} - \boldsymbol{v}_t\cdot\nabla h)$, $w_t \simeq \zeta_{tt} + \boldsymbol{v}_t\cdot\nabla h$.

In connection with the system (3.10), we point out that it is possible to assume $\sigma \ll 1$ independently of the shallow-water condition (3.4). In order to explain the sense of this statement, we seek a solution of (3.10) in the form

$$\xi = \tilde{\xi}\exp[i\boldsymbol{k}\cdot\boldsymbol{x}]\,,$$

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[16] S. C. LOWELL: *Comm. Pure Appl. Math.*, **2**, 275 (1949).

[17] L. LANDAU and E. LIFCHITZ: *Mécanique des fluides* (Moscow, 1971).

[18] The case $h = \infty$ cannot be considered within the framework of the GN model (cf. sect. **4**). We recall that the common approach to this case is based on the assumption $\varepsilon\sigma \ll 1$ (see, *e.g.*, ref. [13], sect. **246**; [17], sect. **12**).

ξ being any of the quantities ζ, $\boldsymbol{v}$, $\mathfrak{p}$, π. By letting $\boldsymbol{k} = k\boldsymbol{n}$, substitution in $(3.10)_{1,2}$ gives

$$\tilde{\xi}_t + h\nabla\cdot\tilde{\boldsymbol{v}} + (\nabla h + i\sigma\boldsymbol{n})\cdot\tilde{\boldsymbol{v}}\,,$$
$$\varrho h^2\tilde{\boldsymbol{v}}_t = -\nabla(h\tilde{\pi}) + p_*\nabla(h\tilde{\xi}) + \tilde{\mathfrak{p}}h\nabla h + (\nabla h - i\sigma\boldsymbol{n})(\tilde{\pi} - p_*\tilde{\xi})\,,$$

while the eqs. $(3.10)_{3,4}$ maintain the same form also for $\tilde{\xi}$. If, according to PEREGRINE [3], we assume $\nabla h = O(1)$, it is a simple matter to see that the set of the quantities $\tilde{\xi}$, like ξ, is solution of the system (3.10).

4. – Conclusions.

The GN model is strongly founded on the assumption that the fluid motion could preserve the *column structure* at every time. Of course, if it is not the case, the GN model no longer holds. This fact poses limitations to the range of applicability of the model. To this purpose, the analysis of a simple physical situation enables us to precise quantitatively the limitations of the model. Let us consider a fluid of uniform depth, $h(x, y) = \bar{h}$, and seek a solution of the system (3.10) in the form of a simple harmonic progressive wave, namely

$$\xi = \xi_0 \exp\left[i(kx - \omega t)\right]. \tag{4.1}$$

Then the system (3.10) becomes an algebraic system for the amplitudes ξ_0, *i.e.*

$$\begin{cases} -i\omega\zeta_0 + i\bar{h}ku_0 = 0\,, \\ -i\omega\varrho\bar{h}u_0 = -ik\pi_0 + ikp_*\zeta_0\,, \\ -\frac{1}{2}\omega^2\varrho\bar{h}\zeta_0 = -\varrho g\zeta_0 + \mathfrak{p}_0\,, \\ -\frac{1}{12}\omega^2\varrho\bar{h}\zeta_0 = \pi_0 - \frac{1}{2}\bar{h}\mathfrak{p}_0 - (p_* + \frac{1}{2}\varrho g\bar{h})\zeta_0\,. \end{cases} \tag{4.2}$$

Besides the system (4.2), the choice (4.1) provides the further result $v_t = 0$; we lose no generality by setting $v \equiv 0$. By straightforward calculations, the system (4.2) allows us to obtain the *dispersion relation*

$$\frac{1}{3}\bar{h}^2\omega^2 + \frac{\omega^2}{k^2} - g\bar{h} = 0\,. \tag{4.3}$$

If we denote by $c_{\mathrm{p}} = \omega/k$ the phase velocity of the wave, the dispersion relation (4.3) yields

$$c_{\mathrm{p}} = c\left(1 - \frac{\bar{h}}{3g}\omega^2\right)^{\frac{1}{2}}, \tag{4.4}$$

where $c = (g\bar{h})^{\frac{1}{2}}$. Moreover, letting $c_g = \partial\omega/\partial k$ be the group velocity, (4.3) gives easily

$$c_g = c\left(1 - \frac{\bar{h}}{3g}\omega^2\right)^{\frac{3}{2}}. \tag{4.5}$$

According to (4.4) and (4.5), in general both c_p and c_g would depend on the angular frequency ω. Nevertheless, within a good degree of accuracy, we have wave propagation without dispersion if the long-wave approximation ($\omega^2 \ll g/\bar{h}$ or $\sigma \ll 1$) holds; in this case

$$c_p = c_g = (g\bar{h})^{\frac{1}{2}}. \tag{4.6}$$

Of course, the result (4.6) holds only under the assumption of horizontal bed. Lastly, we point out that, according to (4.4), (4.5), the phase and the group velocity c_p, c_g are monotonic decreasing functions of ω and vanish for $\omega_c = (3g/\bar{h})^{\frac{1}{2}}$. The critical frequency ω_c is a decreasing function of the depth: for instance $\omega_c = 1$ Hz in corrispondence of $\bar{h} = 30$ m. The GN model is the first theory that we know in which such a behaviour occurs.

The occurrence of a critical frequency shows up the real limitations of the GN model. We notice that the mathematical restrictions obtained are perfectly consistent with the underlying physical model. In fact, while it is acceptable that the column structure may be preserved within the approximation $\sigma \ll 1$ (*i.e.* low frequencies or low depths), on the contrary the column hypothesis is clearly unrealistic in the case $\sigma \gg 1$ (*i.e.* high frequencies or high depths). A confirmation of this viewpoint is given by the well-known result according to which the amplitude of a surface wave is a monotonic decreasing function of the depth (see, *e.g.*, ref. [13], sect. **227**; [17], sect. **12**). In this connection we remark that, if $\sigma \simeq 2.65$, a fluid particle whose depth is half of a wave-length would hardly feel the effect of the surface wave (ref. [10], subsect. **15'17**). In conclusion we can say that, although confined to the case of plane waves propagating in a fluid of constant depth, the preceding analysis reveals the effective limitation for the validity of the GN model: it no longer holds for $\sigma \gg 1$ [19].

It follows from what we have seen so far that the GN model gets its natural setting within the framework of shallow-water theories. Actually, assuming the equality between the hydrostatic and the dynamic pressure, the GN model leads directly to the classical shallow-water theory. On the ground of this observation and by considering the usual theory as an approximate model (see KELLER [14]), the GN model, accounting for a dynamic contribution to the pressure, appears as the natural generalization of the shallow-water theory. Then new and more precise results about propagation of waves in fluids are

[19] As a consequence, the procedures of geometrical optics theory (see, *e.g.*, ref. [2]) cannot be applied to the GN model.

to be expected by applying the theory by GREEN and NAGHDI. Unfortunately, the nonlinearity makes the study of the system (2.10)-(2.13) somewhat hard. Consequently, while looking forwards to examining the general case of finite-amplitude waves, as a first step, we intend to analyse the GN model only within the approximation of small-amplitude waves. This analysis will be developed first by using suitable numerical methods and it will be the subject of a future paper.

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The research reported in this article was carried out in collaboration with the « Laboratorio per la Matematica Applicata - C.N.R. Genova » in connection with the project « Conservazione del suolo », subproject « Dinamica dei Litorali ».

● RIASSUNTO

In questo lavoro si esaminano alcuni modelli per la propagazione delle onde di gravità in acqua di profondità variabile. In tale contesto, particolare attenzione è rivolta al modello proposto recentemente da Green e Naghdi. Un confronto dettagliato con teorie largamente in uso permette di collocare tale modello nell'ambito delle teorie dell'acqua poco profonda. Fortunatamente, la maggior generalità del modello di Green e Naghdi, delineata in questo lavoro, apre nuove prospettive sulla possibilità di ottenere risultati più soddisfacenti per la propagazione di onde di gravità.

Гравитационные волны в воде с переменной толщиной.

Резюме (*). — Эта работа касается анализа некоторых моделей распространения гравитационных волн в воде с переменной толщиной. В рамках этого подхода особое внимание уделяется модели, недавно предложенной Грином и Нагди. Подробное сравнение с общепринятыми теориями позволяет рассмотреть модель Грина и Нагди, как теорию, непосредственно связанную с приближением мелкой воды. Однако обобщение этой модели, предложенное в этой статье, открывает новые возможности получения более удовлетворительных результатов, касающихся распространения гравитационных волн.

(*) *Переведено редакцией.*