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The Shear-Free Condition in Robinson's Theorem[1]

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Abstract

A geometrical interpretation of the shear-free condition, required by Robinson's theorem, is given. In particular it is proved that the shear-free condition for a (geodesic) null congruence is necessary and sufficient in order that the null conditions be preserved along the rays.

It is well known that Robinson's theorem [1] gives a simple characterization of a null electromagnetic field in terms of the associated null congruence. It states explicitly [2, p. 345] that "a congruence of null curves is geodesic and shearfree if and only if the associated family of null bivectors includes a solution of the sourcefree Maxwell's equations."

The physical and geometrical meaning of the geodesic condition has been extensively studied in the literature [3, pp. 343-345]. The main results of this work show that (real) photons travel only along null geodesics. The geometrical and physical role of the shear-free condition, however, is less obvious. The purpose of this note is to show *that the shear-free condition guarantees that the null conditions* [2, p. 273; 3, p. 327]

$$E \cdot H = E^2 - H^2 = 0 \tag{1}$$

are preserved along the null rays.

This result provides a simple scheme that characterizes the different roles played by the geodetic and shear-free conditions (usually treated on the same footing; see, e.g., [2, p. 344] in the formulation of Robinson's theorem. In

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fact, we have the geodesic condition as a *dynamical* consequence of Maxwell's equations [5]; secondly, on a purely *geometrical* ground, the shear-free condition implies the null condition (1).

The formal machinery to prove this interpretation is as follows. Let $\mathcal{F}^{ij} = F^{ij} + i\, {}^*F^{ij}$ be a complex electromagnetic tensor. In the null case, the algebraic structure of $\mathcal{F}^{ij}$ is [2, p. 318, p. 342]

$$\mathcal{F}^{ij} = k^i M^j - k^j M^i \tag{2}$$

with $k^i k_i = k^i M_i = M^i M_i = 0$. Moreover, for every choice of a physical frame of reference [6-9], it is possible to set [10]

$$M = E + iH \tag{3}$$

E, H being, respectively, the relative electric and magnetic fields [5]. A canonical *null tetrad* $\{k, n, m, \overline{m}\}$ [2], associated with the electromagnetic field, is obtained by letting

$$m = M/(M^i \overline{M}_i) \tag{4}$$

and choosing the real null vector n such that equation (A.1) (see Appendix) is identically verified.

We now observe that Maxwell's equation $\mathcal{F}^{ij}{}_{;j} = (k^i M^j - k^j M^i)_{;j} = 0$ may be written in the form

$$[k, M] = \chi k - 2\theta M \tag{5}$$

where [,] indicates the usual Lie bracket, and $\chi = M^i{}_{;i}$, $\theta = \frac{1}{2} k^i{}_{;i}$. Let v be the parameter along the null rays such that $k = \partial/\partial v$; then, setting $\hat{M} \overset{\text{def}}{=} \exp(2\int \theta\, dv) M$, equation (5) reads

$$[k, \hat{M}] = \exp(2\int \theta\, dv) \chi k \tag{6}$$

Comparison of equation (6) with equation (A.3) shows that $\hat{M}$ is a connection vector. If we set

$$\hat{E} \overset{\text{def}}{=} \mathrm{Re}(\hat{M}) = \exp(2\int \theta\, dv) E \qquad \hat{H} \overset{\text{def}}{=} \mathrm{Im}(\hat{M}) = \exp(2\int \theta\, dv) H \tag{7}$$

the null conditions (1) read

$$\hat{E} \cdot \hat{H} = \hat{E}^2 - \hat{H}^2 = 0 \tag{8}$$

Finally, as $\hat{E}$ and $\hat{H}$ are connection vectors, satisfying equation (A.6) by the definition (4), Lemma A.1 implies that equation (8) is preserved along the null rays if and only if the congruence is shear-free.

This result yields the required interpretation.

Appendix

Let $\{k, n, m, \overline{m}\}$ be a null tetrad with k tangent to a null geodesic congruence $\equiv$, parametrized with a parameter v. The only nonzero scalar products are[2]

$$k^i n_i = -m^i \overline{m}_i = 1 \tag{A.1}$$

It is well known that we can expand the quantity $k_{i;j}$ in terms of the null tetrad as follows [2, p. 399; 11]:

$$k_{i;j} = (\gamma + \overline{\gamma})k_i k_j + (\epsilon + \overline{\epsilon})k_i n_j - (\alpha + \overline{\beta})k_i m_j - (\overline{\alpha} + \beta)k_i \overline{m}_j - \overline{\tau} m_i k_j + \overline{\sigma} m_i m_j + \overline{\rho} m_i \overline{m}_j - \tau \overline{m}_i k_j + \rho \overline{m}_i m_j + \sigma \overline{m}_i \overline{m}_j \tag{A.2}$$

where σ is the *shear*, $\rho = -\theta + i\omega$ is the *complex dilatation,* and $\epsilon + \overline{\epsilon}$ is zero if and only if v is the *affine* parameter (notations are as in Ref. [2, 11]). By definition, a *connection vector*[3] ζ satisfies [2, p. 264; 12]

$$[k, \zeta] = fk \tag{A.3}$$

for some function f, and is defined up to a gauge, namely,

$$\zeta \longrightarrow \zeta + gk \tag{A.4}$$

g being a function. Equation (A.3) and the conditions $k^i k_i = 0$, $k^i{}_{;j} k^j = (\epsilon + \overline{\epsilon})k^i$, yield

$$\frac{D}{Dv}(k^i \zeta_i) = (\epsilon + \overline{\epsilon})k^i \zeta_i \tag{A.5}$$

so that $k^i \zeta_i = 0$ for some v implies $k^i \zeta_i = 0$ identically. Finally, it is possible to use the gauge (A.4) to put $\zeta^i n_i = 0$ identically. With these assumptions, we have the following:

Lemma A.1. Let $\{\zeta_{(A)}\}$ $(A = 1, 2)$ be two connection vectors satisfying

$$\zeta^i{}_{(A)} k_i = \zeta^i{}_{(A)} n_i = 0 \tag{A.6}$$

identically. Then the congruence $\equiv$ is *shear-free* if and only if it is possible to define two *mutually orthogonal* differentiable connection vectors with the *same length* along $\equiv$.

Proof. Equation (A.2), (A.3), and (A.6) yield

$$\frac{D}{Dv}(\zeta^i{}_{(A)} \zeta_{(B)i}) = 2(\sigma \overline{m}_i \overline{m}_j + \overline{\sigma} m_i m_j + \rho \overline{m}_i m_j + \overline{\rho} m_i \overline{m}_j) \zeta^{(i}{}_{(A)} \zeta^{j)}{}_{(B)} \tag{A.7}$$

where the parentheses denote symmetrization.

[2] The metric has the signature −2.
[3] For a more detailed discussion of the kinematical and geometrical meaning of connection vectors, see, e.g., [12].

Setting $\zeta_{(A)} = \mu_{(A)} m + \bar{\mu}_{(A)} \bar{m}$, equation (A.7) gives

$$\frac{D}{Dv}(\zeta^i{}_{(1)}\zeta_{(2)i}) = 2(\sigma\mu_{(1)}\mu_{(2)} + \bar{\sigma}\bar{\mu}_{(1)}\bar{\mu}_{(2)}) - 2\theta\zeta^i{}_{(1)}\zeta_{(2)i} \tag{A.8a}$$

$$\frac{D}{Dv}(|\zeta_{(1)}|^2 - |\zeta_{(2)}|^2) = 2\sigma(\mu_{(1)}{}^2 - \mu_{(2)}{}^2) + \bar{\sigma}(\bar{\mu}_{(1)}{}^2 - \bar{\mu}_{(2)}{}^2) - 2\theta(|\zeta_{(1)}|^2 - |\zeta_{(2)}|^2) \tag{A.8b}$$

Let $\equiv$ be shearfree. Fixed any point on a line $a \in \equiv$, we consider two connection vectors such that at P $\zeta^i{}_{(1)}\zeta_{(2)i}|_P = 0$ and $(|\zeta_{(1)}|^2 - |\zeta_{(2)}|^2)_P = 0$. Then eqs. (A.8a, b) imply that these conditions are preserved along the null rays. On the contrary, if one assumes that there exist two mutually orthogonal connection vectors with the same length, the system (A.8a, b) becomes an algebraic system for σ (notice that σ has exactly two degrees of freedom). The condition of linear independence of $\zeta_{(1)}$ and $\zeta_{(2)}$ [namely $\mu_{(1)}\bar{\mu}_{(2)} - \bar{\mu}_{(1)}\mu_{(2)} \neq 0$] implies that the determinant of this algebraic system does not vanish, so we have only the trivial solution $\sigma = 0$. Q.E.D.

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