A Note on Godel's Metric[1]

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Abstract

It is proved that every line element of the form $ds^2 = -dt^2 + dx^2 + dy^2 + D(x)dz^2 - 2E(x)dzdt$, which satisfies Einstein equations for a perfect fluid, is necessarily isometric to the Gödel's universe.

§(1): Introduction

A knowledge of exact solutions of the field equations of general relativity is regarded as important for a better understanding of our real physical universe. However, Einstein's equations also admit unphysical solutions, whose (global) behavior violates some fundamental requirement (time-orientability, strong causality, etc. [1]). Nevertheless, these pathological solutions allow interesting insights into the very features of general relativity from a philosophical point of view [1, 2]. The most familiar example of this type is provided by Gödel's metric [3]. The latter, in fact, contains closed timelike lines, so that this solution is *acausal*. Moreover, matter is everywhere rotating relative to the compass of inertia in contrast with Mach's principle [4].

In this paper we shall analyze Einstein's equations for a perfect fluid assuming the line element in a form that resembles closely the Gödel metric, namely,

$$ds^2 = -dt^2 + dx^2 + dy^2 + D(x)dz^2 - 2E(x)dzdt$$

where $D(x), E(x)$ are a priori unspecified functions of x only.

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Although the previous line element seems "to be of possible use in the construction of more general model universes" (than Gödel's, cf. [5], p. 338), we shall prove that this is not quite true. In fact, a straightforward integration of the Einstein differential system, outlined in Section 2, leads to three different meaningful solutions (of course Gödel's metric is one of them). In Section 3 these solutions are found to be isometric to each other; the links among them are derived explicitly. It is worth noticing that, although one of these isometries is already known, the other one is essentially new. Moreover, this work exhausts all possibilities of writing down Gödel's metric in a Gödel-like fashion.

§(2):   *Field Equations and their Solutions*

Our general assumptions are the following: (a) The mass energy distribution is assumed to be of the form of a perfect fluid, with the energy-momentum tensor [5]

$$T_{ij} = (\mu + p)V_i V_j + pg_{ij} \tag{2.1}$$

where μ is the invariant mass density, p is the pressure, and V_i is the four-velocity of the continuum.[2] (b) The metric tensor is assumed in the form

$$ds^2 = -dt^2 + dx^2 + dy^2 + D(x)dz^2 - 2E(x)dzdt \tag{2.2}$$

where t, x, y, z are *comoving* coordinates markers, without any a priori relation to physically meaningful quantities. The line element (2.2) can also be written thus:

$$ds^2 = -(dt + Edz)^2 + dx^2 + dy^2 + (D + E^2)dz^2$$

which makes it evident that its signature is everywhere +2 if and only if

$$D + E^2 > 0 \tag{2.3}$$

In order to obtain the explicit form of the Einstein field equations, it is convenient to set

$$D + E^2 := \Delta := \exp(\delta) \tag{2.4}$$

The field equations

$$R_{ij} = \chi(T_{ij} - \tfrac{1}{2}Tg_{ij}) \tag{2.5}$$

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[2] Here and in the following, latin indices run from 0 to 3. The use of natural unit ($c = 1$) is implicitly assumed; also we set the cosmological constant Λ to zero, and allow p to be nonzero.

give rise to only four nontrivial relations:

$$\delta_{xx} + \frac{1}{2}\delta_x{}^2 - \frac{1}{\Delta}E_x{}^2 = \chi(p - \mu) \tag{2.6a}$$

$$E_{xx} - \frac{1}{2}E_x\delta_x = 0 \tag{2.6b}$$

$$\frac{1}{\Delta}E_x{}^2 = \chi(\mu + 3p) \tag{2.6c}$$

$$p = \mu \tag{2.6d}$$

where the subscript x indicates partial differentiation with respect to x. A first simplification of the previous system is obtained by replacing equation (2.6d) by its first integral[3]

$$E_x{}^2 = 2k^2\Delta$$

where k is an arbitrary constant and the factor 2 is introduced for convenience. Now, comparison with (2.6c) and (2.6d) gives the compatibility condition of the system (2.6), i.e.,

$$p = \mu = \text{const}$$

This result stands in complete agreement with the behavior of matter in the Gödel solution [5, 6]. Besides, it is worth noticing that, in view of the previous results, the hydrodynamical equations for the fluid, i.e., $T^{ij}{}_{;j} = 0$, are identically satisfied.

We are now ready to write out the field equations (2.6) in a more suitable form:

$$\delta_{xx} + \frac{1}{2}\delta_x{}^2 = 2k^2 \tag{2.7a}$$

$$E_x{}^2 = 2k^2\Delta \tag{2.7b}$$

$$p = \mu = \text{const} \tag{2.7c}$$

$$k^2 = 2\chi\mu \tag{2.7d}$$

In the special case $k = 0$, equations (2.7c) and (2.7d) immediately lead to $p = \mu = 0$ so that the resulting space-time is empty. Moreover, a straightforward calculation gives rise to a vanishing Riemann tensor, implying the flatness of the solution. We remark this is exactly the behavior of Gödel's model when the matter content decreases to zero [4].

Let us now consider the case $k \neq 0$. The differential system (2.7) can be solved explicitly. As this is a routine matter, we only summarize the principal steps of calculation in order to fix notation; details are left to the reader.

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[3]Notice that the other possibility: $E_x{}^2 = -2k^2\Delta$ implies $\Delta \leqslant 0$, which contradicts condition (2.3) and leads to the unphysical conclusion $\mu < 0$ [cf. subsequent equation (2.7d)].

By means of (2.4), equation (2.7a) reads

$$\frac{\Delta_{xx}}{\Delta} - \frac{\Delta_x{}^2}{2\Delta^2} = 2k^2 \qquad (2.8)$$

Integration by quadrature immediately gives

$$\Delta = \tfrac{1}{2}\, a^2\, [\exp\,(kx) - b\,\exp\,(-kx)]^2 \qquad (2.9)$$

where a, b are two arbitrary constants. Equation (2.9) is therefore in complete agreement with (2.3). Actually, equation (2.9) implies $\Delta \geqslant 0$; however, when $\Delta = 0$, one has $(-g)^{1/2} = \Delta^{1/2} = 0$, which gives rise to a coordinate singularity. Inserting this result in equation (2.7b) and integrating, we get

$$E = a\,[\exp\,(kx) + b\,\exp\,(-kx) + n] \qquad (2.10)$$

n being an arbitrary constant.

Finally, equations (2.4), (2.9), and (2.10) yield

$$D = a^2\ \{-\tfrac{1}{2}\,[\exp\,(kx) + b\,\exp\,(-kx)]^2 - 2b - 2n\,[\exp\,(kx)$$
$$+ b\,\exp\,(-kx)] - n^2\}\ (2.11)$$

As was to be expected, when $a = 1, b = n = 0$, the Gödel universe results [3].

§(3): *Discussion and Conclusions*

The problem is now to investigate the dependence of our metric on the choice of the constants a, b, n. The final result is summarized by the following theorem.

Theorem. The solutions of the Einstein equations (2.5) under the conditions (2.1), (2.2) are all isometric to the Gödel universe.

Proof. First of all, it is worth noticing that the constants a, n are indeed inessential. In fact, applying the coordinate transformation

$$t = \tilde{t} + na\tilde{z}, \quad x = \tilde{x}, \quad y = \tilde{y}, \quad z = a\tilde{z}$$

we can simplify the line element to

$$ds^2 = -dt^2 + dx^2 + dy^2 + D(x)dz^2 - 2E(x)dtdz \qquad (3.1)$$

where

$$D = -\tfrac{1}{2}\,[\exp\,(kx) + b\,\exp\,(-kx)]^2 - 2b \qquad (3.2a)$$

$$E = \exp\,(kx) + b\,\exp\,(-kx) \qquad (3.2b)$$

as can also be obtained directly from (2.10), (2.11) by putting $a = 1, n = 0$. Let us now investigate the role of the constant b. Three cases arise.

(i) $b > 0$. Setting

$$x_0 = \frac{1}{2k} \ln b \qquad (3.3)$$

equations (3.2a) and (3.2b) read

$$D = -2 \exp (2kx_0) \, [\cosh^2 k(x - x_0) + 1] \qquad (3.4a)$$

$$E = 2 \exp (kx_0) \cosh k(x - x_0) \qquad (3.4b)$$

so that, by performing the coordinate transformation

$$t = \tilde{t}, \quad x = \tilde{x} - x_0, \quad y = \tilde{y}, \quad z = \exp (kx_0)\tilde{z} \qquad (3.5)$$

the line element is reduced to a particular form which can be obtained from (3.1), (3.2a), and (3.2b) by putting $b = 1$. We have thus proved the following:

> When $b > 0$, the solutions (3.1), (3.2a), and (3.2b) are all isometric to each other.

(ii) $b < 0$. As previously, if we set

$$x_0 = \frac{1}{2k} \ln (-b) \qquad (3.6)$$

equations (3.2a) and (3.2b) read

$$D = -2 \exp (2kx_0) \, [\sinh^2 k(x - x_0) - 1] \qquad (3.7a)$$

$$E = 2 \exp (kx_0) \sinh k(x - x_0) \qquad (3.7b)$$

then, the coordinate transformation (3.5), when x_0 is now defined as in (2.6), leads formally to the line element obtained from (3.1), (3.2a), and (3.2b) by putting $b = -1$. The required result follows at once:

> When $b < 0$, the solutions (3.1), (3.2a), and (3.2b) are all isometric to each other.

(iii) $b = 0$. As we stressed at the end of the previous section, in this case the original form of the Gödel metric results [3]. For the sake of completeness, we emphasize that the coordinate transformation (3.5), where x_0 is now *completely arbitrary*, leaves the form of the line element invariant [3, 5, 6].

In view of the previous results, we have reduced the whole analysis to the study of three different models (namely, $b = 0$, $b = \pm 1$); our goal is therefore reached by exhibiting the isometries among them. To this end, a straightforward (but rather cumbersome) calculation shows that the coordinate transformation[4]

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[4]It goes without saying that we use the coordinate labels $\{t, x, y, z\}$, $\{t', r, y', \phi\}$, $\{\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}\}$ for the $b = 0$, $b = 1$, $b = -1$ models, respectively.

$$\exp{(kx)} = \cosh{(kr)} + \cos{(k\phi/2)}\sinh{(kr)}$$

$$kz \exp{(kx)} = 2^{1/2}\sin{(k\phi/2)}\sinh{(kr)}$$

$$\tan{[k(t - t')/2(2)^{1/2}]} = \exp{(-kr)}\tan{(k\phi/4)}$$

$$y = y'$$

makes the $b = 0$ model into a $b = 1$ model.[5]

Similarly, the coordinate transformation[4]

$$\exp{(kx)} = \sinh{(k\widetilde{x})} + \cosh{(k\widetilde{z}/2)}\cosh{(k\widetilde{x})}$$

$$kz \exp{(kx)} = 2^{1/2}\sinh{(k\widetilde{z}/2)}\cosh{(k\widetilde{x})}$$

$$\tan{[k(t - \widetilde{t})/2(2)^{1/2}]} = -\exp{(-k\widetilde{x})}\tanh{(k\widetilde{z}/4)}$$

$$y = \widetilde{y}$$

makes the $b = 0$ model into a $b = -1$ model. This completes the proof of the Theorem.

References

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[5]We point out that the $b = 1$ model is already known in current literature as it exhibits the rotational behavior of the Gödel universe. Nevertheless, technical reasons suggest choosing the line element by putting $b = 1$, $n = -2$, $a = 1$ in equations (2.10), (2.11) [1, 3].